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Introduction to Special Relativity

Introduction

This lesson is not so much a math lesson as it is a physics lesson. Physics is the study of all physical things, anything and everything in the universe. The way that physics studies the universe is by using math as a tool. Hopefully by the end of this lesson, we'll get a bit of a feeling for what that means exactly. The piece of Physics we'll be looking at is Einstein's theory of Special Relativity.

You may have heard about Special Relativity before. It's one of the 20th century revolutions in physics that really surprises and amazes people. Understanding this theory can change your view entirely on the way the universe works. We won't be going in depth about the theory and its implications. We will focus more on why the theory makes sense, and where it stems from its roots in mathematics and physics. When going through this lesson try focus on these things as well, and not just the "cool stuff" in what the theory means. We want to understand the theory, not just be able to say it.

Classical Physics

To truly understand Special Relativity, we need to understand what came before it. So before moving forward, we're going to take a bit of a step back, and talk a little about classical physics.

In the early 20th century, there were two major theories that created revolutions in physics: the Theory of Relativity, and Quantum Mechanics. Physics before the development of these two theories is often referred to as ***classical physics***. A foundational part of classical physics is ***Newtonian physics***, referring to the work done by or associated with Isaac Newton in the 17th century. Among the most important of his works was the development of ***Newton's Laws of Motion***. We're going to be talking a bit about these first.

Newton's Laws of Motion

There are three "Newton's Laws of Motion", but we're only going to be talking about two of them, the first two, in this lesson. Before that, though, let's talk about some extra terms.

Velocity

is the *vector* for speed. In other words it is a speed + a direciton. For more on vectors see the previous week's lesson, or the Grade 6 Week 1 Fall 2018 Math Circles lesson. The important point about this to keep in mind for this lesson is that for an object to have *constant velocity*, **both** its speed **and** its direction must be constant. So, if an object is moving in a circle, for example, then even if its speed is constant, the velocity will **not be**.

Motion

refers to how something is moving. In particular though, if you're ever asked to describe the motion of something, you're usually being asked about its velocity, and how it might be changing (ie. accelerating).

Newton's First Law of Motion: *The law of inertia*

This law says: Objects will keep their velocity constant, unless they are acted on by an outside force. In other words, objects *resist changes in motion.*

This because of ***inertia***. Inertia is a property of matter that is proportional to mass. The more mass an object has, the more inertia it is. The more inertia it has, the more it *resists* changes in motion. Put simply, it comes down to this: heavier things are harder to push. You can feel this if you ever go shopping. As you put more things in the shopping cart, the mass of the cart increases, and pushing it gets harder. But what is a "push" exactly? What is an outside force?

Newton's Second Law of Motion: *The force law*

Look back at the first law carefully, and notice that "pushing" something means to change its velocity. Newton's Second Law describes this with the following equation:

$$\vec{F} = m\vec{a}$$

In this equation:

- m is the mass of an object in kilograms (kg)
- $\vec{a}$ is the acceleration of this object, in meters per second per second (m/s^2)
- $\vec{F}$ is the *total force* on the object, and is equal to mass × acceleration. Its units are (the units of mass) × (the units of acceleration) = kg×m/s^2. This is usually shortened a new unit, the Newton (N), 1 N = 1 kg×m/s^2

Notice that this force equation doesn't involve velocity at all. A force causes an *acceleration*. Acceleration is a change in velocity, in the same way that velocity is a change in displacement.

This equation also describes the relationship between mass and inertia: more mass with the same amount of force will cause less acceleration, so more mass *resists* changes in motion more.

It's important to keep in mind that force and acceleration are also both *vectors*. So if there is more than one force acting on something, then to find the total force you would need to do a vector addition.

An important thing to understand here is that Newtonian physics is based very heavily on observation. The rules to classical and Newtonian physics fit the real world that we live in amazingly well. It's no wonder that these theories existed for so long, and are even used today. We only needed to use classical physics to get to the moon. The "revolutionary theories" of the 20th century only came about because there were some things that classical physics *could not* explain. These were very particular observations, seen in very particular experiments, that lead to huge developments in science. Science is ultimately limited and guided by the experiments and observations we are able to do.

Principle of Relativity

Newton's 1st and 2nd Laws of Motion above help us arrive at something called the ***Principle of Relativity***. This is still classical physics. The Principle of Relativity is essentially this:

The laws of mechanics are exactly the same for all inertial frames of reference.

There are a few words here that we haven't used yet. Let's unpack them a bit.

- ***Mechanics*** is the part of physics that focuses on things with mass and how they move. This includes Newton's Laws, for example.
- ***Frame of Reference*** means your perspective, or where you're looking from. "Frame of reference" is often shortened to just "frame".
- ***Inertial*** comes from the word inertia, and means going at a constant velocity.

So, Principle of Relativity says that the ways *things with mass* move stay the same in any frame of reference that is *inertial* (or moving with constant velocity, including no velocity). Another way to say this is that:

"Mechanics doesn't care if you're moving or not, what matters is if your frame of reference is inertial."

To relate this to your everyday lives, think about when you're sitting in a car, going straight, with constant speed. When you throw a ball up in the car, it doesn't fly back and hit you in the face. In fact, it behaves in exactly the same way as if you were just standing still. This is a direct result of Newton's Laws of Motion.

Einstein took this concept and extended it to create Special Relativity.

Special Relativity

Special Relativity is based on 2 postulates:

1. The laws of physics are exactly the same for all inertial frames of reference.
2. The speed of light is independent of the motion of the light source.

Every result you may have heard about is then derived from here.

Look carefully at the first point. This is really an extension of the Principle of Relativity, except now it's not only being applied to mechanics. Einstein decided to apply this principle to all of physics. The important addition here is light. Light does not have mass, so it doesn't follow mechanics, or the classical Principle of Relativity. Now this statement essentially says:

"**Nature** doesn't care if you're moving or not, what matters is if your frame is inertial"

Everything that you observe will be exactly the same whether you're moving or not, as long as there is no acceleration. In fact, in a closed room moving with constant velocity, no experiment that you do in this room could detect the room's motion. Everything will be the same as if the room wasn't moving at all. In classical physics an experiment with light would have told you that you're moving, but in Special Relativity it does not.

One result of these postulates is so famous, that it often replaces one of the postulates itself. It's that "the speed of light is always constant *in an inertial frame*". You will often hear this in place of the second postulate, and that's because if both these postulates are true, then it was mathematically shown by Einstein that this result must be true as well.

Time Dilation and Length Contraction

These are the truly world-altering results of Special Relativity. Newtonian physics was entirely based on the observations available at the time. These were all observations that could only be done on a "regular human" scale and speed. So, time and space were both *absolute values* in classical physics, always the same measurable values no matter what frame of reference you put yourself in. Special Relativity changes that. Time and space are no longer absolute values. They now depend entirely on your frame of reference. Essentially, time and space are both *relative*. Nature, in Relativity, is not absolute, but instead only deals with relative quantities. Special Relativity is not a theory about light, it's a theory about the geometry of the world.

There are a lot of online resources and youtube videos that expand on and explain these concepts. A personal favourite video is:
`https://www.youtube.com/watch?v=ev9zrt__lec`
Watch this video and get a feel for how space and time work in Special Relativity. As your speed approaches the speed of light, there are two things that happen: ***time dilation*** and ***length contraction***.

Time Dilation *time slows down*

If Bob is on a space ship, in motion relative to Alice, then from Alice's frame of reference:

$$t_B = \sqrt{1 - \frac{v^2}{c^2}} t_A$$

Where t_A is the amount of time that has passed for Alice, t_B is the amount of time that is passed for Bob *in Alice's frame of reference*, v Bob's speed relative to Alice, and c is the speed of light.

Length Contration *things get squished*

let's say Alive and Bob measure the length of Bob's spaceship before he takes off, and the length is l_A. If Bob is on the space ship, in motion relative to Alice, then from Alice's frame of reference:

$$l_B = \sqrt{1 - \frac{v^2}{c^2}} l_A$$

Where l_A is the length the ship had when it was not moving relative to Alice, and l_B is the length that the ship has now *in Alice's frame of reference.* In Bob's frame of reference, the ship would still have length l_A, and Alice would look squished instead. v is still Bob's speed relative to Alice, and c is the speed of light.

This is completely different from how we're used to thinking about space and time, and all come straight out of the math.

Problems

REVIEW

1. What is Newton's First Law of Motion? Explain the concept of inertia.
   Read the handout above.

2. What is Newton's Second Law of Motion? What is the equation that gives this law?
   Read the handout above. The equation for Newton's Second Law is $\vec{F} = m\vec{a}$

3. Which has more inertia, a ball with mass 10 kg or a ball with mass 20 kg?
   Inertia is proportional to mass, so the ball with higher mass also has more inertia, and so will resist changes in motion more.

4. How is velocity different from speed? Is anything going at a constant speed also going at a constant velocity?
   Velocity is the vector version of speed, so it is a speed + a direction. To go at constant velocity, both your speed AND your direction need to be constant. So something going at a constant speed around a circle, for example, is not going at constant velocity.

5. What is a "frame of reference"? What makes a frame inertial? What kinds of frames does Special Relativity deal with?
   A frame of reference is the perspective or location that things are being observed from. For a frame of reference to be inertial, the frame can must have a constant velocity, relative to all other intertial frames.

6. In classical physics, which of these has an "absolute value" no matter your frame of reference?

   (a) Time
   (b) Space (ie. the length of an object)
   (c) Both (a) and (b)
   (d) Neither (a) nor (b)

   Space and time are both absolute in classical physics, just as we expect them to be from observing our everyday lives.

7. In Special Relativity, which of these has an "absolute value" no matter your frame of reference?

(a) Time

(b) Space (ie. the length of an object)

(c) Both (a) and (b)

(d) Neither (a) nor (b)

Neither space nor time are absolute in Special Relativity. The only absolute quantity is the speed of light in an inertial frame.

8. If Bob is moving at near the speed of light at constant speed relative to Alice, does Alice see his clock as moving faster, slower, or at the same speed as her own? How does Bob see Alices clock?
Alice sees Bob's clock as moving slower due to time dilation. Bob sees Alice's clock as moving slower because from his frame of reference, he's completely still and Alice is moving near the speed of light, so time dilation still applies.

9. If Bob is moving at near the speed of light at constant speed relative to Alice, does Alice see his length as stretched, squished, or the same as if he wasn't moving? How does Bob see Alice?
Alice sees Bob as squished, due to length contraction. Bob sees Alice as squished as well, because from his frame of reference he's completely still and Alice is moving near the speed of light, so length contraction still applies.

APPLY

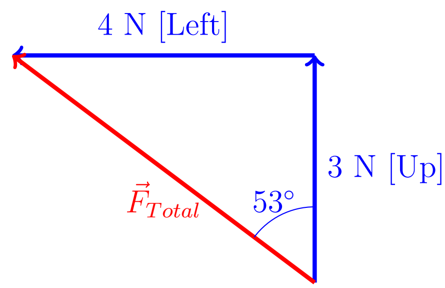
10. What is the total force on an object when

(a) The object has mass 4 kg, and is moving at 3 m/s [Up] without any acceleration?
The total force on this object is 0 N, since there is no acceleration, so no change in the object's motion. Since $\vec{F} = m\vec{a}$, and $\vec{a} = \vec{0}$, then $\vec{F} = \vec{0}$. Note that $\vec{0}$ is "the zero vector", the vector with size 0.

(b) The object has mass 4 kg and is accelerating at 3 m/s^2 [Up]?
In this case we have an acceleration, so we can use the equation $\vec{F} = m\vec{a}$.
$m =$4 kg, and $\vec{a} =$3 m/s^2 [Up], so:

$$\vec{F} = 4 \text{ kg} \times 3 \text{ m/s}^2 \text{ [Up]} = 12 \text{ kg}\times\text{m/s}^2 \text{ [Up]} = 12 \text{ N [Up]}$$

(c) The object has is experiencing a force of 3 N [Up] and 4 N [Left]?
Force is a vector, so the total force must be calculated with a vector addition.



$\vec{F}_{Total} = 5$ N [53° left of verticle] is the total force on the object (rounded to the nearest degree).

11. Do the following calculations on time dilation and length contraction

    (a) If Bob is moving in his space ship at a constant velocity, at a speed of 0.6 times the speed of light ($0.6c$) relative to Alice, and 10 seconds pass for Alice, how much time will she observe has passed for Bob? If 10 seconds pass for Bob, how much time will he obseve has passed for Alice?
    Using the time dilation equation:

$$t_B = \sqrt{1 - \frac{v^2}{c^2}} t_A$$

Where $t_A = 10$ seconds, and $v = 0.6c$ we get:

$$\begin{aligned} t_B &= \sqrt{1 - \frac{(0.6c)^2}{c^2}} \times 10 \text{ seconds} \\ &= \sqrt{1 - 0.36\frac{c^2}{c^2}} \times 10 \text{ seconds} \\ &= \sqrt{1 - 0.36} \times 10 \text{ seconds} \\ &= \sqrt{0.64} \times 10 \text{ seconds} \\ &= 0.8 \times 10 \text{ seconds} \\ t_B &= 8 \text{ seconds} \end{aligned}$$

If 10 seconds pass for Alice, then from her frame of reference she will see that only 8 seconds have passed for Bob. From Bob's frame of reference, Alice is moving at $0.6c$, just in the opposite direction. So, Bob will observe that after 10 seconds pass for him, only 8 seconds have passed for Alice.

(b) If Bob is moving in his space ship at a constant speed of 0.8 times the speed of light ($0.8c$) relative to Alice, and 10 seconds pass for Alice, how much time has passed for Bob? If 10 seconds pass for Bob, how much time has passed for Alice?
Using the time dilation equation:

$$t_B = \sqrt{1 - \frac{v^2}{c^2}} t_A$$

Where $t_A = 10$ seconds, and $v = 0.8c$ we get:

$$\begin{aligned} t_B &= \sqrt{1 - \frac{(0.8c)^2}{c^2}} \times 10 \text{ seconds} \\ &= \sqrt{1 - 0.64\frac{c^2}{c^2}} \times 10 \text{ seconds} \\ &= \sqrt{1 - 0.64} \times 10 \text{ seconds} \\ &= \sqrt{0.36} \times 10 \text{ seconds} \\ &= 0.6 \times 10 \text{ seconds} \\ t_B &= 6 \text{ seconds} \end{aligned}$$

If 10 seconds pass for Alice, then from her frame of reference she will see that only 6 seconds have passed for Bob. From Bob's frame of reference, Alice is moving at $0.8c$, just in the opposite direction. So, Bob will observe that after 10 seconds pass for him, only 6 seconds have passed for Alice.

(c) If Bob is moving in his space ship at a constant speed of 0.6 times the speed of light ($0.6c$) relative to Alice, and Alice measured the length of his space ship to be 20 m while it was not in motion (relative to Alice), what length would Bob's space ship be for Alice now?
Using the length contraction equation:

$$l_B = \sqrt{1 - \frac{v^2}{c^2}} l_A$$

Where $l_A = 20$ m, and $v = 0.6c$ we get:

$$\begin{aligned}
l_B &= \sqrt{1 - \frac{(0.6c)^2}{c^2}} \times 20 \text{ m} \\
&= \sqrt{1 - 0.36\frac{c^2}{c^2}} \times 20 \text{ m} \\
&= \sqrt{1 - 0.36} \times 20 \text{ m} \\
&= \sqrt{0.64} \times 20 \text{ m} \\
&= 0.8 \times 20 \text{ m} \\
l_B &= 16 \text{ m}
\end{aligned}$$

If Alice measured the ship to be 20 m long when it was not moving in her frame of reference, then now that it is moving at $0.6c$ relative to her, she will see the ship as being only 16 m long. In Bob's frame of reference the ship is still 20 m, since it is not moving relative to him, and it's Alice that looks squished instead.