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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

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Symmetry and Music

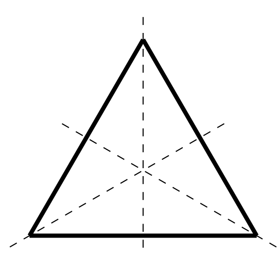
Introduction

We've done a lot of learning and exploring new areas of math this term in Math Circles. For this final lesson, we're going to take a slightly different approach. Today, let's look at something many of you should already be familiar with, learn some more about it, but then focus on ***applications*** of the idea. Let's talk about symmetry!

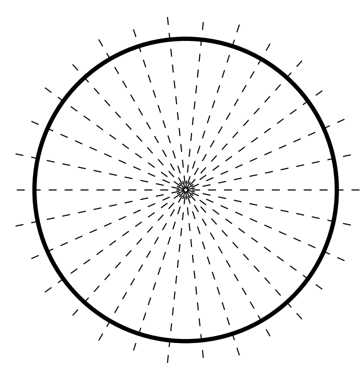
You may have learned about symmetry already in school. You should be most familiar with symmetries in shapes, around something called an axis of symmetry. Let's review this idea, and then learn more!

Symmetry

Symmetry is not usually seen as a particular subject in math, but instead something that comes up over and over again in all areas of math, and in the real world. You should have already learned about symmetries in shapes. Let's look at some examples:



Axis of Symmetry



As you should already know, A triangle as 3 axes of symmetry, while a circle has infinitely many. For a circle, any straight line that goes through its center is an axis of symmetry.

The ***axis of symmetry*** is: A line through the shape such that the shape looks the same on both sides.

At least, that's the symmetry that you're already used to. This type of symmetry that involves drawing an axis through a shape is only one *type* of symmetry:
Reflection Symmetry.

The formal way to say that the triangle has this symmetry is:

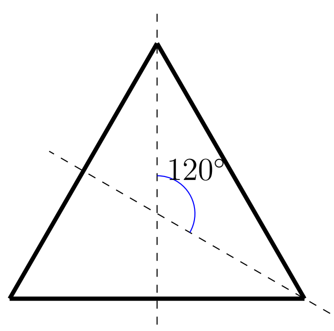
The triangle is ***symmetric*** *with respect to* ***relfections*** about the axis of symmetry.

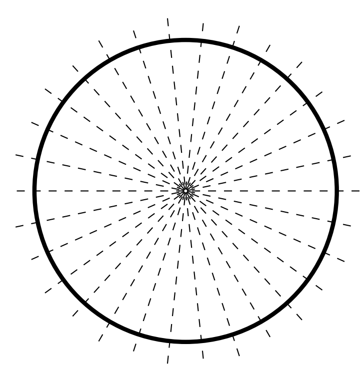
That said, we can of course have other types of symmetry. To understand this, we need to first talk about what a symmetry *really* is. ***Symmetry*** is an "***invariance*** property". This means:

"An object has symmetry if it *doesn't change* after a transformation"

So what the axis of symmetry ***really*** is for a triangle is **not** a line that cuts the triangle in half. An axis of symmetry is a line that we can *reflect* the shape around, that would leave the shape unchanged. The key to knowing if something is symmetric is this: after doing a transformation, does the shape look exactly the same, in exactly the same place? If so, then the shape is *symmetric with respect to* that transformation.

So what other types of symmetries might we have? Well, if symmetry is the property of not changing after a transformation, then let's look at the transformations we know: Reflection, Rotation, and Translation. We've already looked at reflection symmetries. What about the others? Let's look back at the shapes.





An equilateral triangle is symmetric with respect to rotations of 120°, or any multiple of 120°. A circle is symmetric with respect to any rotation. What shapes do you think have translational symmetry? Hint: Think about what symmetry *really* is.

Symmetry and the World

The focus of this lesson, after understanding symmetry, is to look into its applications in the real world.

It's important to note that the rest of this lesson does not cover universally accepted math concepts like we have in Math Circles so far. Instead we're taking the math concept of symmetry and using it to create our own understanding of music and art. This is much closer to being philosophy than math. Doing this is still important, however. Math doesn't happen in an empty world, and thinking about how different subjects interact with each other is an enriching part of any learning.

Symmetry and Beauty

In nature we see symmetry everywhere. From the patterns of butterflies and flowers, to the tiny atoms and molecules that make up everything, we find symmetry in every part of science.

Humans look symmetric on the outside, as much as biology allows us to be. Our symmetric bodies let us walk and interact the way we do with the world around us. Humans like symmetry, and like to find symmetry in world. As beings that are especially good at pattern recognition, we find symmetry pleasing, and easy to take in. In terms of evolution, it's believed that symmetric features were a sign of good health and strength, and so we evolved to recognize it and find it good. That's why in our art, that humans make for themselves, we can still find symmetry, and enjoy it.

To understand how deep our relationship to symmetry is, it might be cool to look for symmetries that are not visual, and see how we can apply the same ideas to them that we did to all the other symmetries so far.

Symmetry and Music

Time Signatures

Let's start with something simple, and work our way up. Listen to sound of a metronome. Notice that the sound repeats consistently every bar. Changing the time signature may change how many beats are in each bar, and how long they are, but the beats would repeat all the same. If we remember that symmetry is an ***invariance property***, we can think of this metronome sound as not changing after a *transformation in time*. This is different from the other symmetries we've talked about so far. All the other symmetries dealt with transformations in space. This symmetry is called ***time-translational symmetry***.

The basic beat pattern given by the time signature unlies all music, and usually stays the same for the whole piece. So the backbone of the music itself is very symmetric in time.

Rhythm

What is a ***rhythm***? Rhythm is a regular repeated pattern. Usually in music it refers more to "length". In other words, a repeated pattern of notes that have a particular length is usually called a rhythm, with less attention to what the notes actually are. Try making your own rhythm, and listen carefully to some of your favourite songs, trying to find the rhythm in them. Unlike the time signature, which the music is built on, the rhythm is a part of the music you can actually hear. It can change sometimes and might be especially different during the chorus for example, but for the most part it repeats throughout the song. This can be understood as another example of time-translational symmetry.

Motif

Talking about rhythm leads us to the concept of motif. In art in general, motif often refers to some dominant idea or feature. In music, it usually refers to a repeating part of the piece that is the "smallest unit of the musical idea". In other words, it while it does repeat in a piece, it **also** conveys the idea of the piece. If you think of music as having a message, a motif is the smallest snippet that still tells you that message. The most famous example of this is in Beethoven's 5th Symphony, which starts with the motif - give it a listen! Unlike the rhythm which repeats throughout a piece, a motif repeats but only sometimes. A motif is less symmetric in time. It has less symmetries, just like a triangle has less symmetries than a circle. This is probably a good thing in terms of the listening experience. Imagine if the motif of Beethoven's 5th symphony repeated for the whole piece. It would be tiring. Can you find any more symmetries in the symphony?

Songs

What are some symmetric features of a song? How does this impact our experience of the song? Listen to some songs you enjoy and think about it!

Try listening to Bohemian Rhapsody by Queen. What do you think of it? Does it sound weird to you? This song is often talked about as being 3 songs put together into one. After listening to it you can probably see why. One way we can think about how this works is by relating it to songs that we're used to hearing. Most songs that we're used to are consistent and full of symmetries. Bohemian Rhapsody is too, but in 3 very distinct parts. When the songs goes from one part to the next, a lot of the symmetries in the music are broken, and rhythms and motifs are changed. **But**, within any of the parts, the music still has all the symmetries that we're used to. This makes it sound as if the song is actually 3 songs put together.

Another very different genre of music where we can find symmetries is rap music. Rap is historically very lyrically driven. Many if not most rap songs have a consistently repeating beat in the background, with lyrics over top. Lyricism in rap is centered around rhyming. This is its own form of symmetry. Analyzing rhyme patterns in rap can reveal complex symmetries with respect to reflections and translations, over instrumentals that are very symmetric in time.

Asymmetries and Music

Symmetries are pleasing and we like them, but they're not always the best thing. If music or art is too symmetric, then it also becomes uninteresting. Humans are good at taking symmetry in all at once. The music doesn't tell a story anymore, it doesn't give us a reason to listen closely, if everything is so symmetric that it always sounds the same. Making good music or art means you have to balance symmetry and repitition, which we like and are important, with ***asymmetries***, or features of the art that are not symmetric. Asymmetry keeps the song interesting, while symmetry keeps it familiar and comfortable. Keeping this balance is something that Bohemain Rhapsody, for example, does really well, in a very interesting way.

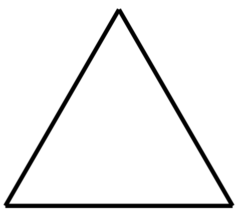
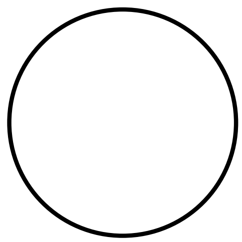
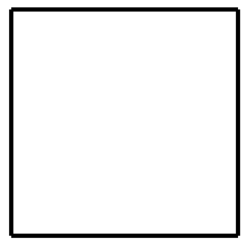
Next time you're listening to music try thinking about this, and think about **why** you enjoy the song.

Problems

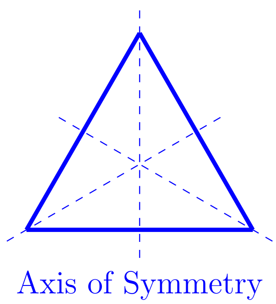
1. What is symmetry? What does it mean for something to be symmetric?

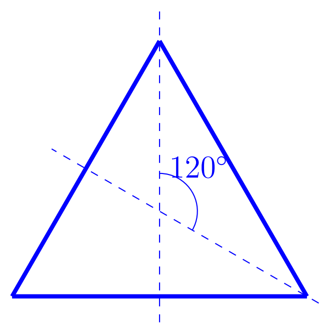
   Symmetry is an invariance property. This means that something has symmetry if it *doesn't change* after a *transformation*. When something has symmetry, then we say that it is *symmetric with respect to the transformation*.

2. Find the symmetries of the following shapes. Make sure to include what type of symmetry it is.

   (a)  (b)  (c) 

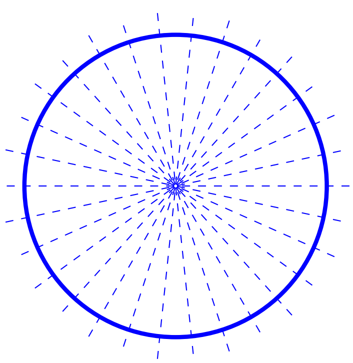
   (a)

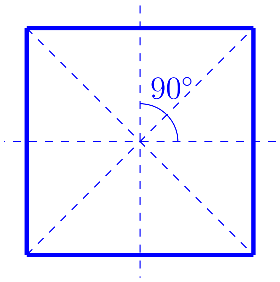
   

   The equilateral triangle, as we've seen, has three reflection symmetries, and is also symmetric with respect to rotations of multiples of 120°.

   (b)

   The circle is symmetric with respect to any rotation, and symmetric with respect to any reflection that has an axis of symmetry going through the center of the circle.

(c) 

A square has 4 axes of symmetry that can be reflected around. The square is symmetric with respect to reflections about any of these axes. It is also symmetric with respect to rotations of multiples of 90°.

3. If you recall back to the coordinate systems lesson, remember that we talked about certain shapes that we "care about" enough to have coordinate systems based around them. Let's explore the symmetries of these shapes. Think about how this might connect to what we talked about in that lesson.

   (a) What are the symmetries of a perfect sphere?
   A perfect sphere, like a circle, is symmetric with respect to any rotation about any line going through its center, and any reflection about any *plane* going through its centre. Note that we're used to talking about symmetries in 2D shapes, where the axis of symmetry is a line, and you rotate about a point. Because we have +1 dimension to the shape now, we also add +1 dimension to the axis of symmetry (making it a plane) and +1 dimension to the rotation point (making it a line).

   (b) What are the symmetries of a cylinder?
   A cylinder is symmetric with respect to any rotation about its **vertical axis**, so that from above it looks like it's just a rotating circle. It is also symmetric with respect to any plane that goes through its **vertical axis**.

   (c) What are the symmetries of an infiitely long cylinder? (ie. a pole that goes on forever)
   An infinitely long cylinder has the same symmetries as a regular cylinder, but it is also symmetric with respect to translations up and down. Because the cylinder goes on forever, "moving" it up and down doesn't actually chage where it is.

   (d) What are the symmetries of a 2D plane? A 2D plane is symmetric with respect to any translation in a direction on the plane, since it goes on forever in all direction in 2 dimensions. It is also symmetric with respect to any rotations about any line *perpendicular* to the plane. We also has reflection symmetries about any plane perpendicular to our plane.

4. Recall that you had a lesson on matrices earlier this term. The ***transpose*** of a matrix is the matrix you get when you take each row and turn it into a column (more details in the matrix lesson). What do you think a matrix would look like if it's *symmetric* with respect to taking the transpose?

   Recall that taking the transpose does the following:

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \qquad \longrightarrow \qquad A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

   For a matrix to be symmetric wirth respect to taking the transpose, it has to *not change* after the transpose is taken. So, a symmetric matrix is always *square*, and has the form:

$$\begin{bmatrix} a & b & c \\ b & e & f \\ c & f & i \end{bmatrix}$$

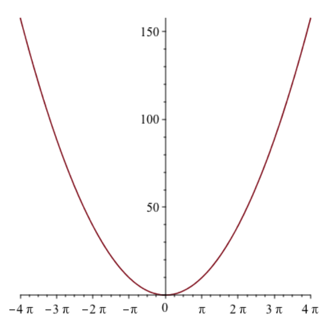
5. As we've learned, we can analyze graphs qualitatively to understand quickly some of the information they convey. One of the the analyses we can do on graphs is to look at if they are ***even*** or ***odd***.

   Even means that the graph is symmetric with respect to reflections about the y-axis.

   Odd means that the graph is symmetric with respect to rotations of 180° about the origin (0,0).

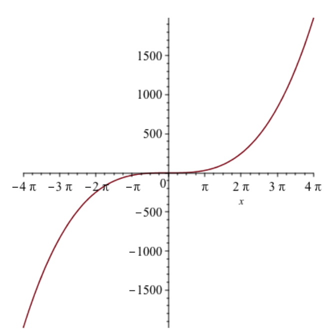
   State whether the following graphs are even or odd.

(a)



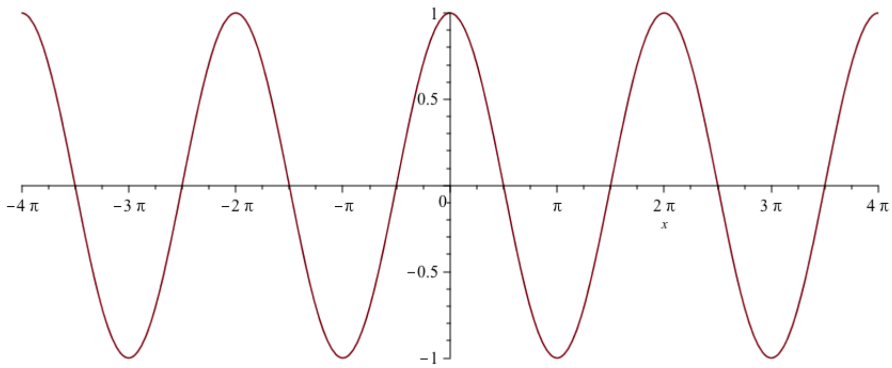
Even

(b)



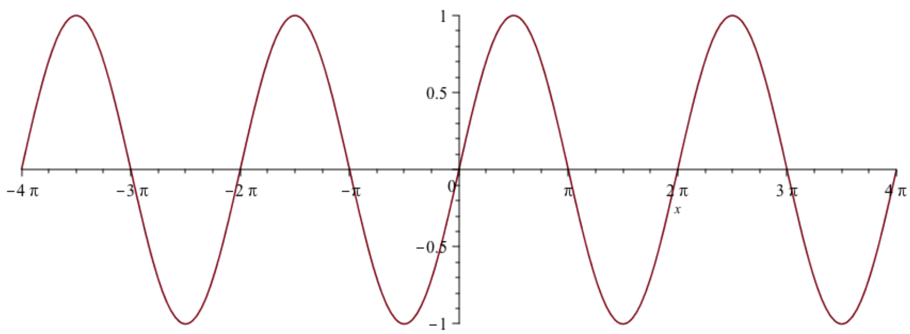
Odd

(c)



Even

(d)



Odd

6. Look for symmetries and asymmetries in your music!