UNIVERSITY OF WATERLOO

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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

Fall 2018 - October 9/10/11

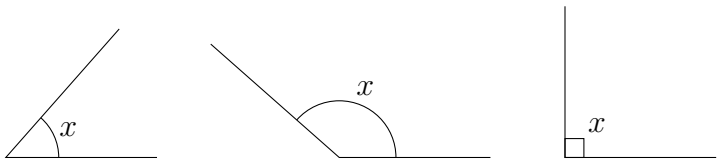
Angles and Light

Today we will be learning about angles. We will learn some angle properties that will allow us to determine the measure of angles without using a protractor. First, we need to recall some basic knowledge about angles.

Angles

The Basics

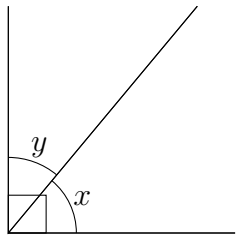
Acute Angle | Obtuse Angle | Right Angle



$x < 90°$ | $x > 90°$ | $x = 90°$

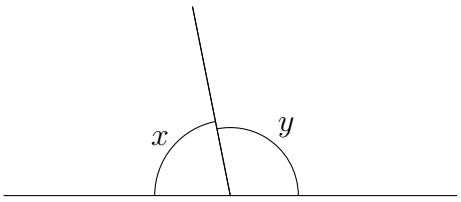
Complementary Angles

Angles are complementary if they add up to 90°



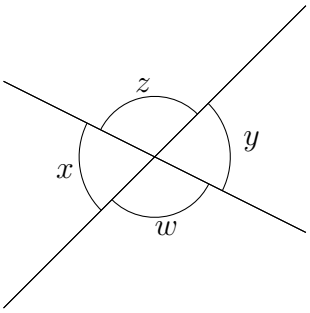
Supplementary Angles

Angles are supplementary if they add up to 180°



Opposite Angles

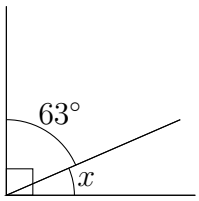
Opposite angles are the angles that are opposite each other when two lines intersect. Using what we know about supplementary angles, what property do opposite angles have?



Solution: Since w and y are supplementary angles and w and x are supplementary angles, we get that x is equal to y. By similar logic, z is equal to w. Therefore, opposite angles are equal.

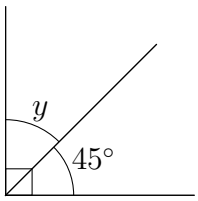
Example 1: Find the unknown complementary angle.

(a)



Solution: $x = 90° - 63° = 27°$

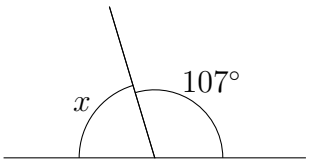
(b)



Solution: $y = 90° - 45° = 45°$

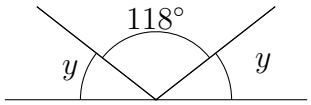
Example 2: Find the unknown angles.

(a)



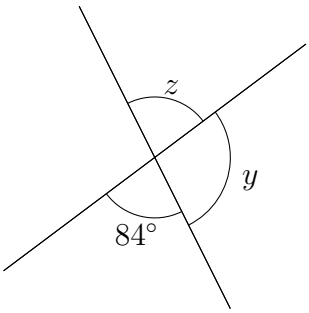
Solution: $x = 180° - 107° = 73°$

(b)



Solution: $118° + 2y = 180°$ so we get $2y = 62°$. Thus, $y = 31°$

(c)



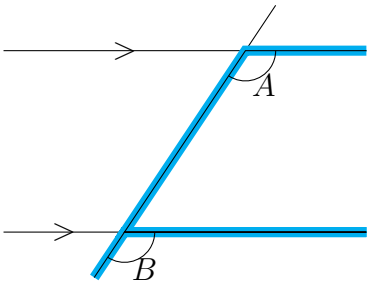
Solution: $y = 180° - 84° = 96°$ by supplementary angles. z is the opposite angle to $84°$ so $z = 84°$

Parallel Lines and Angles

A traversal is a line that intersects two or more lines. If these lines are parallel, the angles around the lines have very nice properties.

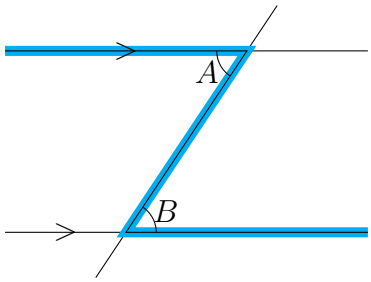
Corresponding Angles

Corresponding angles are angles that are on the same side of the traversal and on the same side of each parallel line. These angles are equal to each other. These types of angles are often referred to as the "F" pattern.



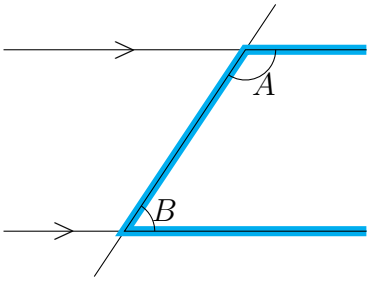
Alternate Angles

Alternate angles are angles that are on opposite sides of the traversal and on opposite sides of the parallel lines. These angles are equal to each other. These types of angles are often referred to as the "Z" pattern.



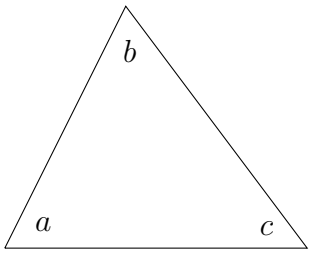
Co-interior Angles

Co-interior angles are angles on the same side of a traversal and on opposite sides of the parallel lines. These angles add up to 180°. This is sometimes referred to as the "C" pattern.



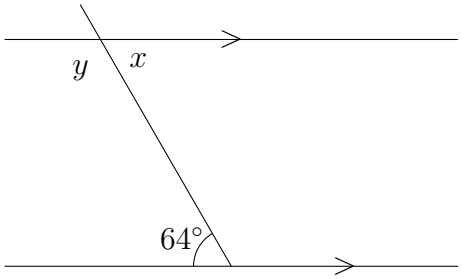
Interior Angles of a Triangle

Triangles have three interior angles. These three angles add up to 180°.



Example 3: Find all the missing angles in the diagrams and state the angle properties you used.

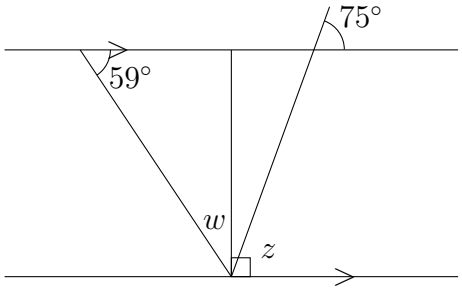
(a)



Solution: Using alternate angles ("Z" pattern), $x = 64°$. Using co-interior angles ("C" pattern) we get that

$$y + 64° = 180° \Rightarrow y = 116°$$

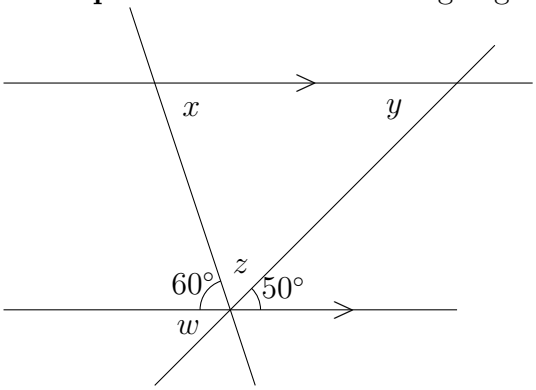
(b)



Solution: By corresponding angles ("F" pattern), $z = 75°$. Then using complementary angles, the angle between w and z is 15°. Now using opposite angles and sum of interior angles of a triangle, we know

$$75° + 59° + w + 15° = 180° \Rightarrow w = 31°$$

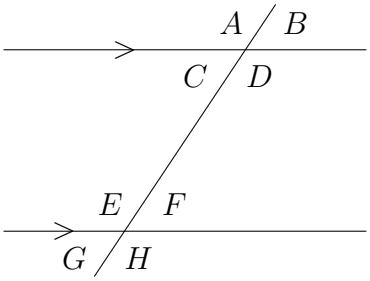
Example 4: Find all the missing angles w, x, y, and z in the diagram using angle properties.



Solution: Using supplementary angles, we know $60° + z + 50° = 180° \Rightarrow z = 70°$ Then, using alternate angles, we see that $y = 50°$. By the sum of the interior angles of a triangle, we get $50° + 70° + x = 180° \Rightarrow x = 60°$. Finally, using opposite angles, $w = 50°$

Thinking Question

How many angles need to be known in order to determine all the angles in the diagram?



Solution: Without loss of generality, let us assume we know the measure of $\angle A$. Then since $\angle A$ and $\angle B$ are supplementary angles we can find out the measure of $\angle B$. Now we can find out the measures of $\angle C$ and $\angle D$ by using their opposite angles. At this point we know $\angle A$, $\angle B$, $\angle C$, and $\angle D$. Now we use alternate angles (the "Z" pattern) with $\angle C$ to get the measure of $\angle F$. Similarly, we use alternate angles with $\angle D$ to get the measure of $\angle E$. Then we can use corresponding angles ("F" pattern) with $\angle C$ to get the measure of $\angle G$. Finally, we use corresponding angles ("F" pattern) with $\angle D$ to get the measure of $\angle H$. Therefore, we have shown that you only need to know the measure of one angle in order to determine all other angles in the diagram.

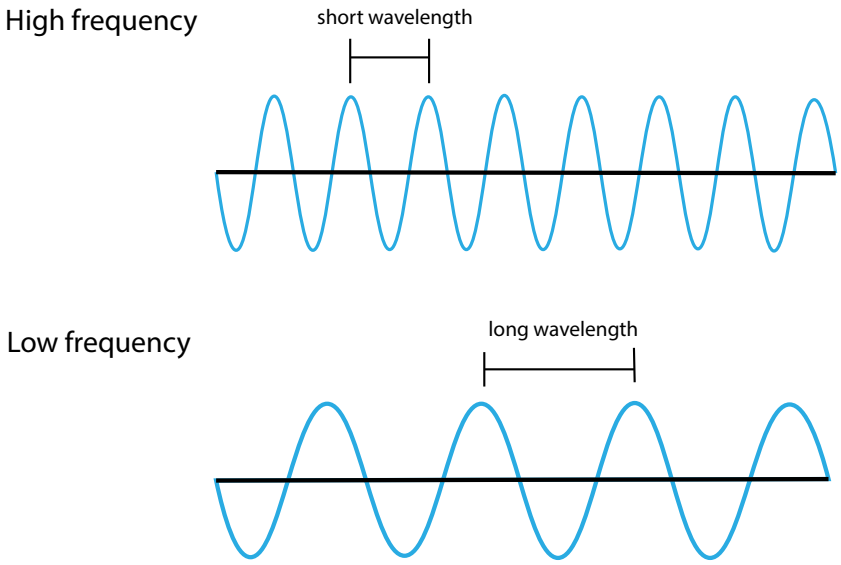
Light

Like sound, light travels in a wave. The light that we can see is only a small part of the electromagnetic spectrum. The electromagnetic spectrum is organized by frequency and wavelength. **Frequency** is the number of waves in a unit of time. **Wavelength** is the distance between corresponding points of two consecutive waves. This is, it is the distance it takes for the wave's shape to repeat.

Wavelength and frequency are inversely related. The higher the frequency, the shorter the wavelength. Also, the longer a wavelength, the lower the frequency. The relationship between wavelength and frequency also involves the speed of light and can be represented by the equation

$$c = \lambda f$$

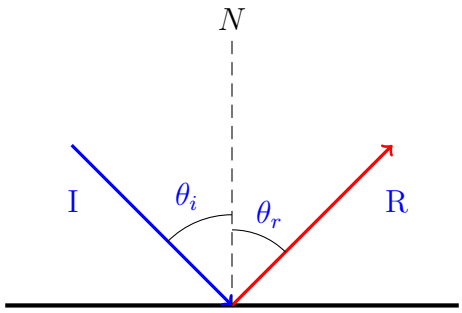
where c is the speed of light, λ is the wavelength and f is the frequency.



To represent light in diagrams, we use a line with an arrow in the direction the light is heading. You can picture this as the beam of light that comes from a flashlight or laser. On diagrams, we also draw the normal. **The normal** is the direction perpendicular (at right angles) to a surface. It is drawn as a dashed perpendicular line at the point where the beam of light hits the surface.

What happens when a ray of light hits a flat mirror?

Solution:



incident ray (I): the ray of light headed towards the mirror

reflected ray (R): the ray of light headed away the mirror

angle of incidence (θ_i): the angle between the normal and the incident ray

angle of reflection (θ_r): the angle between the normal and the reflected ray

point of incidence: the point where the ray of light hits the mirror

The Law of Reflection states that: **when a ray of light reflects off a surface, the angle of incidence is equal to the angle of reflection**

Example 5: A ray of light hits a flat mirror. It strikes the mirror with an angle of incidence of 40°.

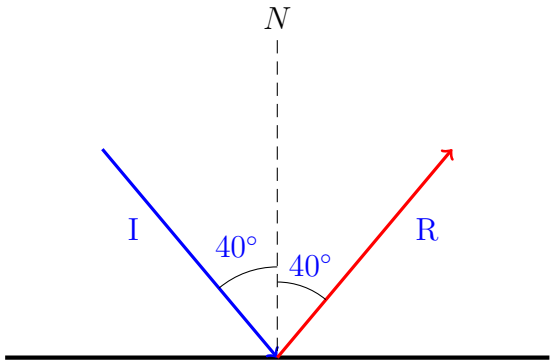
(a) What is the angle of reflection?

(b) Draw a diagram to represent the situation. Label the normal (N), the incident ray (I), and the reflected ray (R).

Solution:

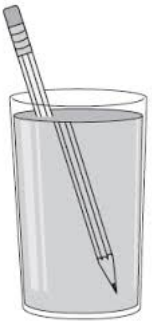
(a) By the Law of Reflection, the angle of incidence is the same as the angle of reflection. We are given that the angle of incidence is 40° so the angle of incidence is 40°.

(b)



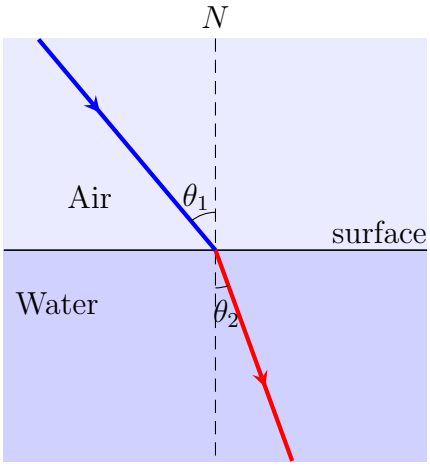
What happens when light travels from air to water?

When a pencil is placed in a cup of water on an angle, the pencil looks as if it has been broken into two pieces. This "Broken Pencil" phenomenon can be explained by refraction.

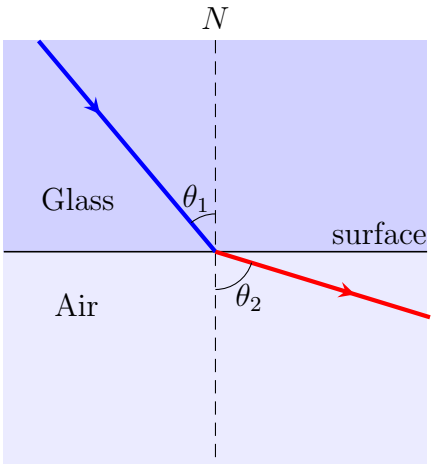


Retrieved from http://www.solpass.org/science4-5/light/print/5-3-light-released-questions-key.pdf?section=study-9

Refraction is the bending of light as it crosses the boundary separating two media. The light bends because it changes speed, either speeding up or slowing down depending on the material.



Light refracts towards the normal if the light wave passes from a medium where it travels fast to a medium where it travels slow (a medium that is more optically dense).



Light refracts away from the normal if the light wave passes from a medium in which it travels slow to a medium where it travels fast (a medium that is less optically dense).

How do we know how optically dense a medium is?

To determine the optically density of a medium, we use the refractive index (or index of refraction). This is the speed of light in a vacuum divided by the speed of the light in the medium. The greater the refractive index, the slower light travels in that medium. For example, the refractive index of water is 1.333 which means light travels 1.333 times slower in water than it does in air. Refractive indices of common media are stored in a table below.

Medium	Refractive Index (n)
Vacuum	1
Ice	1.31
Water	1.333
Olive Oil	1.47
Halite (rock salt)	1.516
Diamond	2.42

Example 6: Given two different media, state which one light travels faster in.

(a) Water and Diamond

Solution: Water

(b) Olive Oil and Ice

Solution: Ice

(c) Water and Ice

Solution: Ice

(d) Air and Halite

Solution: Air

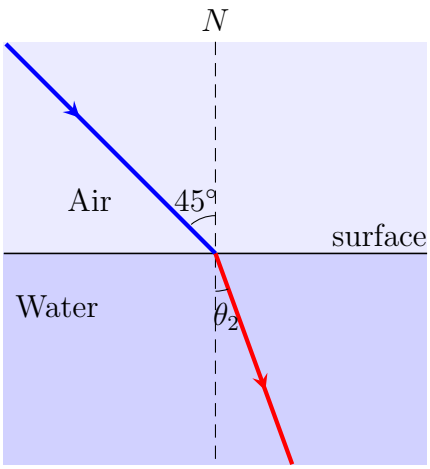
(e) Diamond and Olive Oil

Solution: Olive Oil

Can we use the refractive index to determine the angle of refraction (θ_2)?
We can determine the angle of refraction (how much the light bends as it enters a new medium) by using Snell's Law. **Snell's Law** is stated as

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where n_1 is the refractive index of medium 1 (the medium that the light ray starts in), θ_1 is the angle of incidence, n_2 is the refractive index of medium 2 (the medium that the light ray is refracted into), and θ_2 is the angle of refraction.



Let's use Snell's law on a specific case. We will use the example of the "broken pencil" and say that light is traveling from air to water with an angle of incidence of 45°.

Using the information we've been given, we get

$$1 \sin(45°) = 1.333 \sin \theta_2$$

(remember we assume air has the same refractive index as a vacuum). This equation can be solved. However, it can't be solved with the knowledge we have now. You will learn how to solve equations like this in high school. The solution is $\theta_2 = 32°$. This means that the angle of refraction is 32°.

Problem Set

1. (a) What is the complementary angle of 57°?

   Solution: 33°

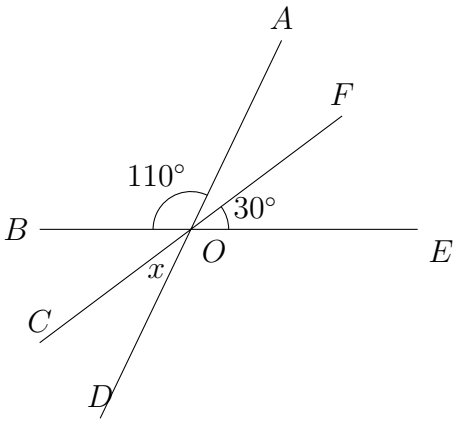
   (b) What is the supplementary angle of 95°?

   Solution: 85°

   (c) What is the opposite angle to 112°?

   Solution: 112°

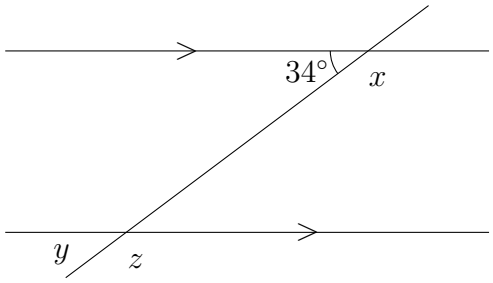
2. In the diagram, what is the value of x? *(Problems, Problems, Problems, Volume 6: page 40, question 4)*



Solution: Since $\angle BOC$ and $\angle FOE$ are opposite angles, $\angle BOC = 30°$. Then using supplementary angles, $\angle AOB + \angle BOC + \angle COD = 180°$. Filling in what we know, $110° + 30° + x = 180°$. Hence, $x = 40°$.

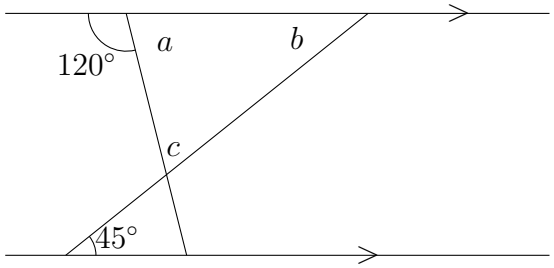
3. Find all the missing angles of the following diagrams and explain your reasoning.

   (a)



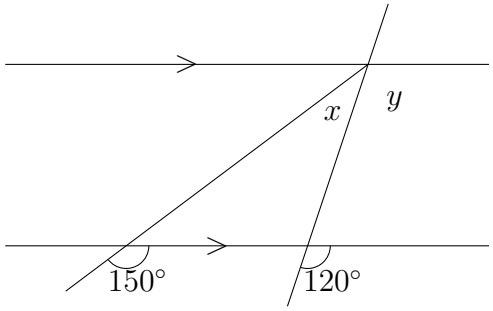
Solution: Since x is a supplementary angle of 34°, $x = 146°$. Then x and z are corresponding angles ("F" pattern), so $z = x = 146°$. Then y is a corresponding angle to 34° so $y = 34°$.

(b)



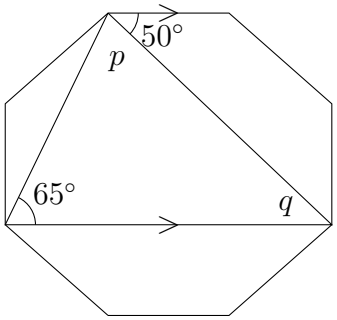
Solution: Since a is a supplementary angle of 120°, $x = 60°$. Then b and 45° are alternate angles ("Z" pattern), so $b = 45°$. Using the sum of interior angles of a triangle we get that $c = 180° - 60° - 45°$.

(c)



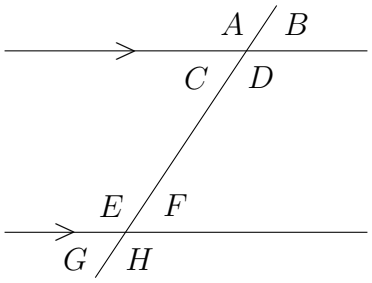
Solution: y is a corresponding angle ("F" pattern) with 120° so, $y = 120°$. Then $x + y$ and 150° are corresponding angles ("F" pattern), so we get that $x + 120° = 150°$. We get $x = 30°$.

(d)



Solution: $p + 50°$ and 65° are co-interior angles ("C" pattern) which means that they sum to 180°. So we get, $p + 65° + 50° = 180°$ which gives $p = 65°$. Then q and 50° are alternate angles ("Z" pattern), so we get the $q = 50°$.

4. List all pairs of angles that are supplementary to each other in the diagram.



Solution: (A,B), (A,C), (A,G), (A,F), (B,D), (B,H), (B,E), (C,D), (C,E), (C,H), (D,F), (D,G), (E,F), (E,G), (F,H), (G,H)

5. A ray of light is reflected off a flat mirror. The light is reflected off the mirror at an angle of 15° from the normal.

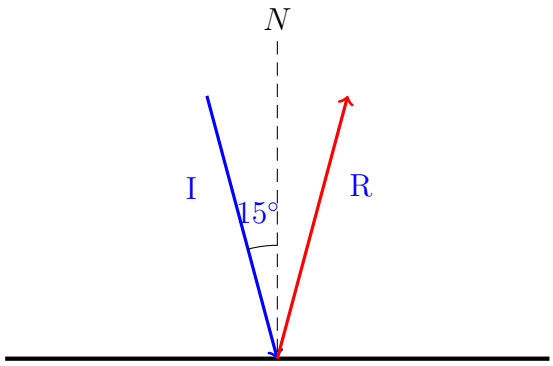
   (a) What is the angle of incidence?

   (b) Draw a diagram to represent the situation. Label the normal (N), the incident ray (I), and the reflected ray (R).

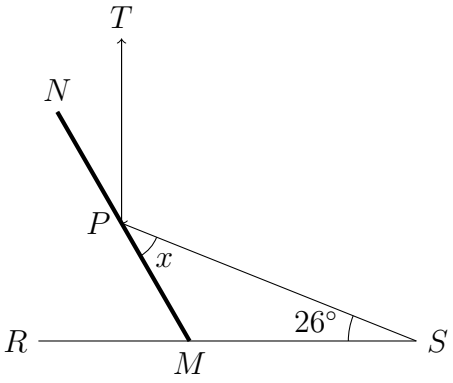
   Solution:

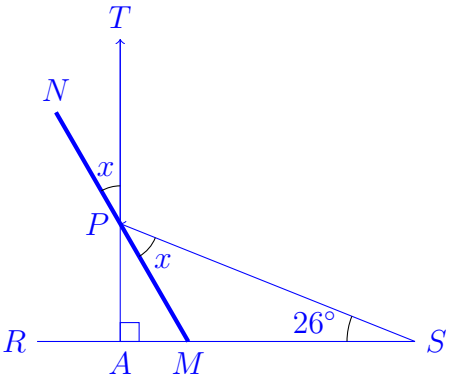
   (a) By the Law of Reflection, the angle of incidence is the same as the angle of reflection. We are given that the angle of reflection is 15° so the angle of incidence is 15°.

   (b)



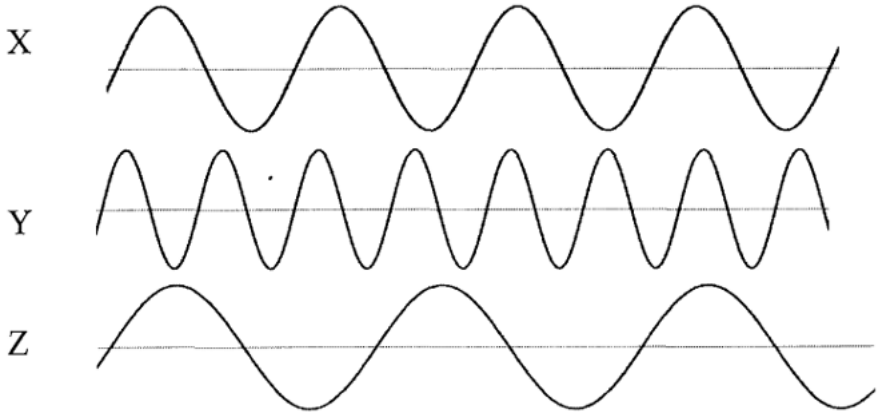
6. A beam of light shines from point S, reflects off a mirror MN at point P, and reaches point T so that PT is perpendicular to RS. What is the measure of $\angle SPM$? *(Problems, Problems, Problems, Volume 7: page 37, question 10)* **Hint:** Think about the Law of Reflection





Solution:
In the diagram, extend TP to meet RS at A. Since $AT \perp RS$, then $\angle SPA = 180° - 90° - 26° = 64°$. Then by the Law of Reflection, $\angle TPN = \angle SPM$. Since, $\angle TPN$ and $\angle MPA$ are opposite angles, they are equal, so $\angle MPA = x$. Then $\angle SPA = 2x = 64°$. Dividing both sides by 2 gives $x = 32°$. Therefore, $\angle SPM$ is 32°.

7. List the waves (X, Y, and Z) in order from highest frequency to lowest frequency.



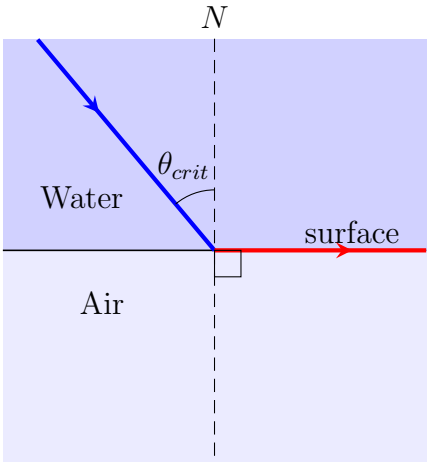
Solution: Y, X, Z

8. Sometimes the incident ray does not refract when it hits the boundary between two media. Instead, it is reflected back away from the boundary. This is called **total internal reflection** and occurs only if the following two statements are both true:

   (i) the light is coming from a more optically dense medium towards a less optically dense medium

(ii) the angle of incidence is greater than the critical angle

The **critical angle** (θ_{crit}) is the greatest angle at which light can strike the boundary between the two media without undergoing total internal reflection (being totally reflected into the first medium). For example, the critical angle from water to air is 48.6°. Given the critical angle and two media, state whether total internal reflection or refraction will occur and explain your reasoning.



Note: To determine the relative optical densities of the media, use the table of refractive indices given in the lesson.

(a) Light travels from a diamond towards air at an angle of 16° from the normal. The critical angle is 24.4°.

Solution: The light starts in a diamond which has a refractive index of 2.42 which means it is very optically dense (light travels slower in it). Air has an approximate refractive index of 1 so it is less optically dense (light travels fast in it). So, the first statement for total internal reflection is true. The angle of incidence is 16° which is less than the critical angle of 24.4. Thus, the second statement is not true. So refraction occurs.

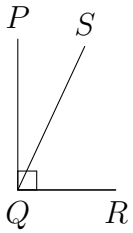
(b) Light travels from water towards air at an angle of 54° from the normal. The critical angle is 48.6°.

Solution: The light starts in water which has a refractive index of 1.333 which means it is more optically dense than air which has a refractive index of 1. This is because light travels slower in water than it does in air. So, the first statement for total internal reflection is true. The angle of incidence is 54° which is greater than the critical angle of 48.6. Thus, the second statement is true. Since both statements are true, total internal reflection occurs.

(c) Light travels from ice to halite (the salt used to melt ice) at an angle of 42° from the normal. The critical angle is 59.8°.

Solution: The light starts in ice which has a refractive index of 1.31 which means it is less optically dense than halite which has a refractive index of 1.516. This is because light travels faster in ice than it does in halite. So, the first statement for total internal reflection is not true. We don't need to check the critical angle to know that refraction will occur instead of total internal reflection.

9. * In the diagram, $\angle PQR$ is 90°, and $\angle RQS$ is 50° greater than $\angle PQS$. What is the measure of $\angle PQS$? *(Problems, Problems, Problems, Volume 1: page 5, question 2)*



Solution: If $\angle PQS$ is x, then $\angle SQR$ is $x + 50°$. Therefore we get $x + x + 50° = 90°$ which results in $x = 20°$. So, $\angle PQS = 20°$.

10. ** When total internal reflection occurs and light is reflected back when it hits the boundary, do you think the angle of incidence will be equal to the angle of reflection? Why or why not? Use the knowledge gained from the lesson and the description of total internal reflection in question 8.

Solution: Note: This is a thinking question and answers may differ. However, students should reflect on learned concepts and justify their thinking.

I think that the angle of incidence will be equal to the angle of reflection because the Law of Reflection states that when a ray of light reflects off a surface, the angle of incidence is equal to the angle of reflection and when the statements to make total internal reflection occur are true, the second medium behaves as a surface. The second medium behaves like a surface because it does not allow any light to pass through it.