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Grade 7/8 Math Circles

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Matrices

Today we will be learning about a very useful mathematical tool called a matrix. A **matrix** is a rectangular arrangement of numbers. In this lesson we will learn how to set up a matrix or several matrices to represent a problem. We will also learn how to apply some operations that you already know to matrices in a way that you haven't seen before. The main purpose of a matrix is to represent and organize information without labels.

For example, suppose you and a friend each have a box of Smarties and you each want to count all your red and blue Smarties. We can represent this information in a table and then convert it into a matrix as shown below.

	Red	Blue
You	5	4
Friend	3	6

$$\Longrightarrow \begin{bmatrix} 5 & 4 \\ 3 & 6 \end{bmatrix}$$

Example 1: Suppose Garden A has 4 cabbages and 11 potatoes and Garden B has 10 cabbages and 7 potatoes. Represent this information in a matrix.

Solution:

	Cabbages	Potatoes
Garden A	4	11
Garden B	10	7

$$\Longrightarrow \begin{bmatrix} 4 & 11 \\ 10 & 7 \end{bmatrix}$$

Some terms to understand

The **dimensions** of matrix are how many rows and columns it has. So far we have only seen 2×2 matrices. In general, an $m \times n$ has m rows and n columns. Matrix A (shown below) is a 2×3 matrix since there are two rows and three columns.

$$A = \underbrace{\begin{bmatrix} 5 & 1 & 6 \\ 2 & 4 & 3 \end{bmatrix}}_{\text{3 columns}} \Big\} \text{ 2 rows}$$

If a matrix has m rows and m columns, it is an $m \times m$ matrix or a **square matrix**. The matrix below is an example of a 2×2 matrix which is a square matrix.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Example 2: What are the dimensions of the following matrices?

(a) $\begin{bmatrix} 3 & 1 \\ 2 & 8 \\ 4 & 6 \end{bmatrix}$

Solution: 3×2 matrix

(b) $\begin{bmatrix} 7 & 4 & 5 \\ 4 & 2 & 3 \\ 7 & 2 & 1 \end{bmatrix}$

Solution: 3×3 matrix

The numbers inside a matrix are called **elements**. To generalize a matrix, we can label each element in our matrices using variables. For example,

$$\begin{array}{ccccc} 2 \times 2 & 3 \times 2 & 2 \times 3 & 3 \times 3 & 3 \times 1 \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} & \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} & \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} & \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{array}$$

To refer to a certain element in a matrix we use indexing. To talk about the element in the second row and third column in matrix A we would say $a_{2,3}$. For example, in the matrix below, $a_{1,2}$ refers to 5 because it is the element in the first row and the second column.

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 4 \\ 6 & 2 \end{bmatrix}$$

Matrices in Real-life

Matrices are used in many different fields and professions. Here are a few examples:

- *Cryptography* - Matrices can be used to encrypt and decrypt secret messages. The sender encrypts the message using an encoding matrix and the receiver decrypts it using a decoding matrix.

$$Sender \Longrightarrow \overset{\text{encoding matrix}}{\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}} \overset{\text{message}}{\begin{bmatrix} 6 & 3 \\ 1 & 7 \end{bmatrix}} = \overset{\text{encrypted message}}{\begin{bmatrix} 33 & 36 \\ 20 & 23 \end{bmatrix}}$$

$$Receiver \Longrightarrow \overset{\text{decoding matrix}}{\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}} \overset{\text{encrypted message}}{\begin{bmatrix} 33 & 36 \\ 20 & 23 \end{bmatrix}} = \overset{\text{message}}{\begin{bmatrix} 6 & 3 \\ 1 & 7 \end{bmatrix}}$$

- *Economics* - Businesses use matrices to study stock market trends to earn a larger profit and to lower their losses

$$\begin{array}{ccccccc} & Revenue & & Costs & & Profit \\ November & \begin{bmatrix} 125 & 55 & 240 \\ 125 & 80 & 200 \end{bmatrix} & - & \begin{bmatrix} 50 & 75 & 60 \\ 40 & 70 & 60 \end{bmatrix} & = & \begin{bmatrix} 75 & -20 & 180 \\ 85 & 10 & 140 \end{bmatrix} \end{array}$$

- *Google* - Ever wonder how Google always knows what you search for? Or wonder how they determine what sites to direct you to? Google uses a page ranking method that uses matrices!

Matrix Operations

Since matrices generally have more than one element, matrix operations do not follow the same rules that operations between two numbers do. They have their own way of performing the usual operations of addition, subtraction, and multiplication.

Matrix Addition

To perform matrix addition, we add together elements that are in the same position in each matrix. Suppose we want to add two 2×2 matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

When we add two matrices together, do they need to have the same dimensions?

Solution: Yes because we are adding elements which have the same index (are located in the same spot in the matrices).

Example 3: Add the following matrices.

$$\begin{bmatrix} 11 & 3 & 5 \\ -1 & 8 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 8 & 6 \\ 5 & 10 & 2 \end{bmatrix} =$$

Solution: $\begin{bmatrix} 13 & 11 & 11 \\ 4 & 18 & 6 \end{bmatrix}$

Example 4: In the 2018 Winter Olympics, Canada won 11 gold medals, 8 silver medals, and 10 bronze medals. Norway won 14 gold, 14 silver, and 11 bronze. China won 1 gold medal, 6 silver medals, and 2 bronze medals.

In the 2014 Winter Olympics, Canada won 10 gold medals, 10 silver medals, and 5 bronze. Norway won 11 gold, 5 silver, and 10 bronze while China won 3 gold, 4 silver, and 2 bronze. How many of each medal does each country have from the past two winter olympics?

Solution:

$$\begin{bmatrix} 11 & 8 & 10 \\ 14 & 14 & 11 \\ 1 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 10 & 10 & 5 \\ 11 & 5 & 10 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 21 & 18 & 15 \\ 25 & 19 & 21 \\ 4 & 10 & 4 \end{bmatrix}$$

From the past two winter olympics, Canada won 21 gold medals, 18 silver medals, and 15 medals, Norway won 25 gold medals, 19 silver medals, and 21 bronze medals, and China won 4 gold medals, 10 silver medals, and 4 bronze medals.

Matrix Subtraction

Matrix subtraction works the same as addition in that we subtract elements that are in the same position in each matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Like matrix addition, matrices need to have the same dimensions in order to perform matrix subtraction.

Example 5: Perform the following matrix subtraction.

$$\begin{bmatrix} 11 & 8 \\ 7 & 24 \\ 6 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 0 & 15 \\ 9 & 2 \end{bmatrix} =$$

Solution: $\begin{bmatrix} 9 & 3 \\ 7 & 9 \\ -3 & 1 \end{bmatrix}$

Example 6: Tina and Louise get a weekly allowance of \$5 and they are competing to see who can save the most money until Wednesday.

- On Monday, Tina and Louise do not spend any money
- On Tuesday, Tina buys a snack at the vending machine for \$1.75
- On Wednesday, Louise loses \$2 in a bet with her friends and Tina spend \$1.30 on a chocolate bar

On Wednesday who saved the most amount of money? Tina or Louise?

Solution:

$$\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 1.75 \\ 0 \end{bmatrix} - \begin{bmatrix} 1.30 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.95 \\ 3 \end{bmatrix}$$

By Wednesday, Tina has \$1.95 and Louise has \$3.00, so Louise saved the most money.

Scalar Multiplication

A **scalar** is a number. To perform scalar multiplication, we multiply each element in the matrix by the same number.

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Example 7: Calculate the following:

(a) $4 \begin{bmatrix} 1 & 10 & 3 \\ 6 & 1 & 2 \\ 5 & 7 & 4 \end{bmatrix} =$

Solution: $\begin{bmatrix} 4 & 40 & 12 \\ 24 & 4 & 8 \\ 20 & 28 & 16 \end{bmatrix}$

(b) $3 \begin{bmatrix} 1 & 6 \\ 10 & -3 \end{bmatrix} =$

Solution: $\begin{bmatrix} 3 & 18 \\ 30 & -9 \end{bmatrix}$

Example 8: Yesterday at the Elmira Maple Syrup Festival, there are were 45 kids, 12 teenagers, and 51 adults who went on the carriage ride and 31 kids, 55 teenagers, and 75 adults who went on the guided tour. Today, there were twice as many people because word got out about how fun the festival was. How many kids, teenagers, and adults went on the carriage ride? How many took a guided tour?

Solution:

$$2 \begin{bmatrix} 45 & 12 & 51 \\ 31 & 55 & 75 \end{bmatrix} = \begin{bmatrix} 90 & 24 & 102 \\ 62 & 110 & 150 \end{bmatrix}$$

Today, 90 kids, 24 teenagers, and 102 adults went on the carriage ride and 62 kids, 110 teenagers, and 150 adults took the guided tour.

Matrix Multiplication

Like the title suggests, matrix multiplication involves multiplying two matrices together. To perform matrix multiplication, we multiply each element of a row from one matrix by the corresponding element of the column from the other matrix and add the product together. Below, see what happens when we multiply two 2×2 matrices together. For a way to visualize matrix multiplication, watch this youtube video https://www.youtube.com/watch?v=bFeM4ICRt0M

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Suppose A is an $m \times n$ matrix and B is a $p \times q$ matrix. Matrices A and B can only be multiplied together (like AB) if $\underline{\quad n = p \text{ (\# of columns in } A = \text{ \# rows in } B) \quad}$

Example 9: Calculate the following.

(a) $\begin{bmatrix} 2 & 7 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$

Solution: $\begin{bmatrix} 2(3)+7(4) & 2(1)+7(6) \\ 5(3)+3(4) & 5(1)+3(6) \end{bmatrix} = \begin{bmatrix} 34 & 44 \\ 27 & 23 \end{bmatrix}$

(b) $\begin{bmatrix} 7 & 1 \\ 2 & 5 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 5 & 3 & 4 \end{bmatrix}$

Solution: $\begin{bmatrix} 7(3)+1(5) & 7(2)+1(3) & 7(1)+1(4) \\ 2(3)+5(5) & 2(2)+5(3) & 2(1)+5(4) \\ 10(3)+4(5) & 10(2)+4(3) & 10(1)+4(4) \end{bmatrix} = \begin{bmatrix} 26 & 17 & 11 \\ 31 & 19 & 22 \\ 50 & 32 & 26 \end{bmatrix}$

Thinking Question

When you multiply two matrices together is the result always a square matrix? If yes, explain why. If no, give a example of matrix multiplication that results in a matrix that isn't square.

Example 10: Alain has 1 loonie, 2 quarters, 7 dimes, and 10 nickels. Yoshi has 2 loonies, 1 quarter, 8 dimes, and 6 nickels. Who has more money? Use matrices to figure out the answer.

Solution: $\begin{bmatrix} 1 & 2 & 7 & 10 \\ 2 & 1 & 8 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0.25 \\ 0.10 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 1 + 0.50 + 0.70 + 0.50 \\ 2 + 0.25 + 0.80 + 0.30 \end{bmatrix} = \begin{bmatrix} 2.70 \\ 3.35 \end{bmatrix}$

Yoshi has more money.

Transpose

The **transpose** is an operator which switches the row and column indices of the original matrix, A, and produces another matrix, A^T, which has rows that used to be the columns of A. For example,

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \qquad A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

Example 11: Determine A^T.

(a) $A = \begin{bmatrix} 11 & 1 \\ 5 & 14 \\ 6 & 2 \end{bmatrix}$

Solution: $A^T = \begin{bmatrix} 11 & 5 & 6 \\ 1 & 14 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 7 \\ 9 & 13 \end{bmatrix}$

Solution: $A^T = \begin{bmatrix} 2 & 9 \\ 7 & 13 \end{bmatrix}$

(c) $A = \begin{bmatrix} 6 & 8 & 5 & 1 \\ 3 & 10 & 9 & 8 \\ 7 & 2 & 16 & 4 \end{bmatrix}$

Solution: $A^T = \begin{bmatrix} 6 & 3 & 7 \\ 8 & 10 & 2 \\ 5 & 9 & 16 \\ 1 & 8 & 4 \end{bmatrix}$

Applications of Matrices

The Area of a Parallelogram

Matrices can be used to represent points on a grid. For example, if we were to represent the point (4,3) with a matrix we would write it as

$$p = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

This representation can be generalized as a point $u = (x, y)$ written as

$$u = \begin{bmatrix} x \\ y \end{bmatrix}$$

This 2×1 matrix is called a vector. Vectors include all $n \times 1$ representations of points in space. If we are given two points that represent opposite points on a parallelogram and we know that (0,0) is the bottom left corner of the parallelogram, we can calculate the area of the parallelogram. We do this by taking the absolute value of the **determinant** of the 2×2 matrix formed by putting the two points together. The absolute value of a number is the number but *positive*. For example, the absolute value of 6 is 6 and the absolute value of -7 is 7.

First, let's look how to calculate the determinant of a 2×2 matrix.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Example 12: Calculate the determinant of the matrix $\begin{bmatrix} 2 & 6 \\ 7 & 1 \end{bmatrix}$

Solution: $\det \begin{bmatrix} 2 & 1 \\ 7 & 6 \end{bmatrix} = 2(6) - 1(7) = 5$

Now we can find the area of a parallelogram using the below formula

$$\text{Area} = \left| \det \begin{bmatrix} p & q \end{bmatrix} \right|$$

Example 13: Find the area of the parallelograms using the points given.

(a) $p = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ & $q = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

Solution: $\text{Area} = \left| \det \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} \right| = |4(5) - 2(3)| = |12| = 12$

The area of the parallelogram is 12 units^2

(b) $u = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$ & $v = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$

Solution: $\text{Area} = \left| \det \begin{bmatrix} 6 & 10 \\ 8 & 4 \end{bmatrix} \right| = |6(4) - 8(10)| = |-56| = 56$

The area of the parallelogram is 56 units^2

Problem Set

1. Determine whether the following operations are possible. If they are, state the dimensions of the produced matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \qquad C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad D = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad E = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

   (a) $A - C$

   Solution: No it is not possible.

   (b) $A + A$

   Solution: Yes it results in a 2×2 matrix.

   (c) BE

   Solution: No it is not possible.

   (d) $2D$

   Solution: Yes it results in a 3×3 matrix

   (e) DE

   Solution: Yes it results in a 3×1 matrix

   (f) CB

   Solution: Yes it results in a 2×2 matrix

   (g) $B - C^T$

   Solution: Yes it results in a 3×2 matrix

   (h) $BC + A$

   Solution: No it is not possible.

   (i) $C^T A$

   Solution: Yes it results in a 3×2 matrix

2. Add or subtract the following matrices:

$$A = \begin{bmatrix} 10 & 5 & 6 \\ 1 & 17 & 20 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 14 & 37 \\ 51 & 8 & 23 \end{bmatrix} \qquad C = \begin{bmatrix} 11 & 2 & 7 \\ 18 & 15 & 3 \end{bmatrix}$$

(a) $A + B$

Solution: $\begin{bmatrix} 19 & 19 & 43 \\ 52 & 25 & 43 \end{bmatrix}$

(c) $A + B - C$

Solution: $\begin{bmatrix} 8 & 17 & 36 \\ 24 & 10 & 40 \end{bmatrix}$

(b) $B - C$

Solution: $\begin{bmatrix} -2 & 12 & 30 \\ 33 & 7 & 20 \end{bmatrix}$

(d) $B + C - A$

Solution: $\begin{bmatrix} 10 & 11 & 38 \\ 68 & 6 & 6 \end{bmatrix}$

3. Multiply the following matrices:

$$D = \begin{bmatrix} 7 & 0 & 10 \\ 15 & 9 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} 3 & 11 \\ 8 & 5 \\ 1 & 20 \end{bmatrix} \qquad F = \begin{bmatrix} 6 & 12 \\ 7 & 4 \end{bmatrix}$$

(a) DE

Solution: $\begin{bmatrix} 31 & 277 \\ 119 & 250 \end{bmatrix}$

(b) EF

Solution: $\begin{bmatrix} 95 & 80 \\ 83 & 116 \\ 146 & 92 \end{bmatrix}$

(c) FD

Solution: $\begin{bmatrix} 222 & 108 & 84 \\ 109 & 36 & 78 \end{bmatrix}$

4. Calculate the following given matrices G and H:

$$G = \begin{bmatrix} 1 & 3 \\ 9 & 4 \end{bmatrix} \qquad H = \begin{bmatrix} 8 & 3 \\ 2 & 5 \end{bmatrix}$$

(a) $3G + H$

Solution: $\begin{bmatrix} 11 & 12 \\ 29 & 17 \end{bmatrix}$

(b) $5G - 4H$

Solution: $\begin{bmatrix} -27 & 3 \\ 37 & 0 \end{bmatrix}$

5. Monique, Akiko, and Martin are playing Monopoly. They each start off with only \$1500. In the beginning they all have zero properties, and zero houses.

   In the first trip around the board:

   - Monique spends \$750 and gets 2 properties
   - Akiko buys one property for \$600
   - Martin buys 3 properties for \$1000 but he lands on Akiko's property and has to pay her \$20

   In the second trip around the board:

   - They each get \$200 for passing GO
   - Monique buys another property for \$125. She also buys 3 houses for \$600
   - Akiko buys one property for \$450
   - Martin also makes a deal with Akiko where he gives her \$500 in exchange for one of her properties. He also buys 1 house for \$75

   (a) How much money does each player have after the first trip around the board? How many properties and houses does each player have after the first trip around the board? Using matrices and matrix operations to find the answer.

   (b) After the second trip around the board how much money and how many properties and houses does each player have? Use matrices and matrix operations and your answer from part (a).

   Solution: A different number of matrices and matrix operations can be used to solve this problem. Below is only one way of many that we can represent the problem.

   (a)

$$1^{st}\text{ Trip } \begin{bmatrix} 1500 & 0 & 0 \\ 1500 & 0 & 0 \\ 1500 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 750 & 0 & 0 \\ 600 & 0 & 0 \\ 1020 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 750 & 2 & 0 \\ 900 & 1 & 0 \\ 480 & 3 & 0 \end{bmatrix}$$

Monique has \$750, two properties and no houses. Akiko has \$900, one property, and no houses. Martin has \$480, three properties, and no houses.

(b)

$$2^{nd}\text{ Trip}\begin{bmatrix}750 & 2 & 0\\900 & 1 & 0\\480 & 3 & 0\end{bmatrix}+\begin{bmatrix}200 & 0 & 0\\200 & 0 & 0\\200 & 0 & 0\end{bmatrix}-\begin{bmatrix}725 & 0 & 0\\450 & 0 & 0\\575 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 1 & 3\\500 & 1 & 0\\0 & 1 & 1\end{bmatrix}-\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}225 & 3 & 3\\1150 & 1 & 0\\105 & 4 & 1\end{bmatrix}$$

Monique has \$225, three properties and three houses. Akiko has \$1150, one property, and no houses. Martin has \$105, four properties, and one house.

6. When multiplying numbers, we can say $2 \times 3 = 3 \times 2 = 6$. In general,

$$a \times b = b \times a$$

where a and b is any number. This is called the **commutative property** of numbers. Is this property true for matrices? Answer the following questions to find out. Suppose we have the following matrices:

$$A=\begin{bmatrix}5 & 6 & 2\\10 & 4 & 1\end{bmatrix} \qquad B=\begin{bmatrix}2 & 3\\8 & 5\\1 & 1\end{bmatrix}$$

(a) Calculate the following:

(i) AB

Solution:

$$AB=\begin{bmatrix}60 & 47\\53 & 51\end{bmatrix}$$

(ii) BA

Solution:

$$BA=\begin{bmatrix}40 & 24 & 7\\90 & 68 & 21\\15 & 10 & 3\end{bmatrix}$$

(b) Does $AB = BA$? Is the commutative property true for matrices?

Solution: No. $AB \neq BA$ because matrix AB has different dimensions than matrix BA. Thus, the commutative property is **not** true for matrices.

7. (a) * Find the 2×2 matrix A which, when multiplied with any 2×2 matrix B, gives an answer of B.

   Solution: If $AB = B$ then $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This matrix is called the identity matrix.

   (b) Using what you learned from part (a), what is the 3×3 matrix A which, when multiplied with any 3×3 matrix B, gives an answer of B?

   Solution: Using part (a) we can extend the pattern to get $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. ****Hill Cipher** (Cryptography)
   Suppose we number the alphabet from 0 to 25. (i.e. $A = 0, B = 1, C = 2, ...$). As an example, consider the message `FRIEND` as a 3×2 matrix and the 3×3 *key matrix* (or encoding matrix) below:

$$\texttt{FRIEND} \Rightarrow \begin{bmatrix} \texttt{F} & \texttt{E} \\ \texttt{R} & \texttt{N} \\ \texttt{I} & \texttt{D} \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 4 \\ 17 & 13 \\ 8 & 3 \end{bmatrix} \qquad \text{Key matrix} \Rightarrow \begin{bmatrix} 6 & 12 & 1 \\ 15 & 7 & 2 \\ 8 & 13 & 3 \end{bmatrix}$$

   To encrypt the message, multiply the key matrix by the message. For the resulting matrix, subtract 26 from each element repeatedly until the element is less than 26.

$$\begin{bmatrix} 6 & 12 & 1 \\ 15 & 7 & 2 \\ 8 & 13 & 3 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 17 & 13 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 242 & 183 \\ 210 & 157 \\ 285 & 210 \end{bmatrix} \Rightarrow \begin{bmatrix} 8 & 1 \\ 2 & 1 \\ 25 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} \texttt{I} & \texttt{B} \\ \texttt{C} & \texttt{B} \\ \texttt{Z} & \texttt{C} \end{bmatrix} \Rightarrow \texttt{ICZBBC}$$

   The encrypted message is `ICZBBC`. To decrpyt a message, you need another key matrix. Multiply this key matrix by the encrypted message matrix to determine the actual message. (Note that all elements in every matrix must be a number from 0 to 25.)

(a) Encrypt the following message given the key matrix: `HAPPINESS`

$$\text{Key matrix} \Rightarrow \begin{bmatrix} 2 & 10 & 5 \\ 21 & 4 & 6 \\ 11 & 8 & 3 \end{bmatrix}$$

Solution:

$$\texttt{HAPPINESS} \Rightarrow \begin{bmatrix} 7 & 15 & 4 \\ 0 & 8 & 18 \\ 15 & 13 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 10 & 5 \\ 21 & 4 & 6 \\ 11 & 8 & 3 \end{bmatrix} \begin{bmatrix} 7 & 15 & 4 \\ 0 & 8 & 18 \\ 15 & 13 & 18 \end{bmatrix} = \begin{bmatrix} 89 & 175 & 278 \\ 237 & 425 & 264 \\ 122 & 268 & 242 \end{bmatrix} \Rightarrow \begin{bmatrix} 11 & 19 & 18 \\ 3 & 9 & 4 \\ 18 & 8 & 8 \end{bmatrix} \Rightarrow \texttt{LDSTJISEI}$$

$$\texttt{HAPPINESS} \Longrightarrow \texttt{LDSTJISEI}$$

(b) Decrypt the following encrypted message given the key matrix: `YKVJ`

$$\text{Key matrix} \Rightarrow \begin{bmatrix} 8 & 21 \\ 23 & 2 \end{bmatrix}$$

Solution:

$$\texttt{YKVJ} \Rightarrow \begin{bmatrix} 24 & 21 \\ 10 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 21 \\ 23 & 2 \end{bmatrix} \begin{bmatrix} 24 & 21 \\ 10 & 9 \end{bmatrix} = \begin{bmatrix} 402 & 357 \\ 572 & 501 \end{bmatrix} \Rightarrow \begin{bmatrix} 12 & 19 \\ 0 & 7 \end{bmatrix} \Rightarrow \texttt{MATH}$$

$$\texttt{YKVJ} \Longrightarrow \texttt{MATH}$$