Exercise 1.
(1) If $a, b$ are coprime positive integers and $ab = c^2$ for some integer $c$, show that $a = t^2$ and $b = s^2$ for some integers $t$ and $s$.
(2) Show that for any integer $x$ the numbers $x$ and $x^2 + 1$ are coprime.
(3) Numbers $0, 1, 2^2 = 4, 3^2 = 9, \ldots$ are called squares. Show that the distance between $k^2$ and $(k + 1)^2$ is equal to $2k + 1$. When is this distance equal to 1?
(4) Use the previous results to conclude that the equation $y^2 = x^3 + x$ has no solutions in positive integers $x$ and $y$.

1. Gaussian Integers

Exercise 2. Let $a + bi, c + di$ be Gaussian integers. Prove the following:
(1) Every rational integer is a Gaussian integer;
(2) $(a + bi) + (c + di)$ is a Gaussian integer;
(3) $(a + bi) - (c + di)$ is a Gaussian integer;
(4) $(a + bi)(c + di)$ is a Gaussian integer.

Exercise 3. Prove that $1 + 2i$ divides 5 but does not divide 7.

Exercise 4. Let $\alpha, \beta$ be Gaussian integers. Prove the following:
(1) $N(\alpha\beta) = N(\alpha)N(\beta)$; Show that $N(\alpha) \geq 0$ for all Gaussian integers and $N(\alpha) = 0$ if and only if $\alpha = 0$. Thus the norm is non-negative.

Exercise 5.
(1) Show that if $\alpha$ is a Gaussian unit then $N(\alpha) = 1$.
(2) Prove that the units of $\mathbb{Z}[i]$ are 1, $-1$, $i$ and $-i$.

Exercise 6. Find Gaussian primes among the integers 2, 3, 5, 7.
2. Sums of Two Squares

In this exercise we will investigate which numbers \( n \) can be written as the sum of two squares. That is, \( n = a^2 + b^2 \) for some integers \( a \) and \( b \).

**Exercise.** Compute first 10 numbers that are sums of two squares.

**Step 1.** Let \( m \) and \( n \) be positive integers that are sums of two squares. Prove that \( mn \) is also a sum of two squares. **Hint:** use the fact that the norm \( N \) is multiplicative.

**Step 2.** Prove that every integer that is a sum of two squares is of the form \( 4k, 4k + 1 \) or \( 4k + 2 \) for some integer \( k \). Conclude that every rational prime \( p \) of the form \( 4k + 3 \) is not a sum of two squares, and so it is a Gaussian prime.

**Step 3.** Let \( p \) be a rational prime of the form \( 4k + 1 \). In this exercise, we will use the fact that there always exists an integer \( x \) such that \( p \mid x^2 + 1 \).

1. Show that \( p \) does not divide neither \( x + i \) nor \( x - i \). Conclude that it is not prime, so \( p = \alpha \beta \) for some Gaussian integers \( \alpha, \beta \).
2. Prove that neither \( \alpha \) nor \( \beta \) are units. Conclude that \( N(\alpha) = p \), so \( p \) is a sum of two squares.

**Step 4.** Show that 2 is a sum of 2 squares. Conclude that every number of the form

\[
2^t p_1^{e_1} \cdots p_k^{e_k} q_1^{2f_1} \cdots q_l^{2f_l}
\]

is a sum of two squares, where \( p_i \) are primes of the form \( 4k + 1 \) and \( q_i \) are primes of the form \( 4k + 3 \).