Intermediate Math Circles
Wednesday March 02, 2016
Introduction to Vectors II

1. If $\vec{u} = [2, 3]$, $\vec{v} = [5, 1]$, and $\vec{w} = [1, -4]$, find the following:

   (a) i. $\vec{u} \cdot \vec{v}$

   $$[2, 3] \cdot [5, 1] = (-3)(5) + (2)(1) = -13$$

   ii. $\vec{v} \cdot \vec{u}$

   $$[5, 1] \cdot [2, 3] = (5)(-3) + (1)(2) = -13$$

   iii. $\hat{u}$

   $$\frac{1}{\sqrt{13}}[2, 3]$$

   iv. $(\vec{u} + \vec{v}) \cdot \vec{w}$

   $$([2, 3] + [5, 1]) \cdot [1, -4] = [2, 3] \cdot [1, -4] = (2)(1) + (3)(-4) = 2 - 12 = -10$$

   v. $(3\vec{u}) \cdot \vec{v}$

   $$(3[2, 3]) \cdot [5, 1] = [-6, 9] \cdot [5, 1] = -45 + 6 = -39$$

   vi. $3(\vec{u} \cdot \vec{v})$

   $$3(-13) = -39$$

   vii. $\vec{w} \cdot \vec{w}$

   $$[1, -4] \cdot [1, -4] = (1)(1) + (-4)(-4) = 17$$

   viii. $\vec{u} \times \vec{v}$

   $$[2, 3] \times [5, 1] = (-3)(1) - (2)(5) = -3 - 10 = -13$$

   ix. $\vec{v} \times \vec{w}$

   $$[5, 1] \times [1, -4] = (5)(-4) - (1)(1) = -21$$

   x. $\vec{w} \times \vec{u}$

   $$[1, -4] \times [2, 3] = (1)(2) - (-4)(-3) = 2 - 12 = -10$$

   (b) Find $d(\vec{u}, \vec{v})$, $d(\vec{v}, \vec{w})$, and $d(\vec{w}, \vec{u})$.

   $$d(\vec{u}, \vec{v}) = |\vec{u} - \vec{v}| = \sqrt{(-3 - 5)^2 + (2 - 1)^2} = \sqrt{65}$$

$$d(\vec{v},\vec{w}) = |\vec{v}-\vec{w}| = \sqrt{(5-1)^2+(1-(-4))^2} = \sqrt{41}$$
$$d(\vec{w},\vec{u}) = |\vec{w}-\vec{u}| = \sqrt{(1-(-3))^2+(-4-2)^2} = \sqrt{52}$$

(c) Find the area of the triangle formed by $\vec{u}$ and $\vec{v}$ using the formula

$$|\vec{u}\times\vec{v}| = |\vec{u}|\;|\vec{v}|\sqrt{1-(\hat{u}\cdot\hat{v})^2}$$

We have that

$$|\vec{u}| = \sqrt{13} \implies \hat{u} = \frac{1}{\sqrt{13}}[2,3]$$
$$|\vec{v}| = \sqrt{26} \implies \hat{v} = \frac{1}{\sqrt{26}}[5,1]$$

Therefore,

$$\hat{u}\cdot\hat{v} = \frac{1}{\sqrt{13}}\frac{1}{\sqrt{26}}[(-3)(5)+(2)(1)] = \frac{-13}{\sqrt{338}}$$
$$\implies (\hat{u}\cdot\hat{v})^2 = \left(\frac{-13}{\sqrt{338}}\right)^2 = \frac{169}{338} = \frac{1}{2}$$

Completing our solution

$$\begin{aligned}|\vec{u}\times\vec{v}| &= \sqrt{13}\sqrt{26}\sqrt{1-\frac{1}{2}}\\ &= \sqrt{338}\sqrt{\frac{1}{2}}\\ &= \frac{\sqrt{338}}{\sqrt{2}}\\ &= \sqrt{\frac{338}{2}}\\ &= \sqrt{169}\\ &= 13\end{aligned}$$

2. Let $\vec{x} = [-2, -5]$ and $\vec{y} = [k, 4]$, what value of k makes these vectors perpendicular?

   We need $\vec{x} \cdot \vec{y} = 0$. Therefore, $\vec{x} \cdot \vec{y} = (-2)(k) + (-5)(4) = 0$.

$$\begin{aligned} -2k - 20 &= 0 \\ 2k &= -20 \\ k &= -10 \end{aligned}$$

3. Let $\vec{x} = [3k + 2, -3]$ and $\vec{y} = [6, 5]$, what value of k makes these vectors perpendicular?

   We need $\vec{x} \cdot \vec{y} = 0$. Therefore, $\vec{x} \cdot \vec{y} = (3k + 2)(6) + (-3)(5) = 0$.

$$\begin{aligned} 18k + 12 - 15 &= 0 \\ 18k &= 3 \\ k &= \frac{3}{18} = \frac{1}{6} \end{aligned}$$

4. (a) The two direction vectors are $\vec{m}_1 = [2, -1]$ and $\vec{m}_2 = [-4, 2]$.
   Since $\vec{m}_2 = -2[2, -1] = 2\vec{m}_1$, then the direction vectors are parallel and hence the lines are parallel.

   (b) The two direction vectors are $\vec{m}_1 = [-2, 3]$ and $\vec{m}_2 = [3, 2]$.
   Now $\vec{m}_1 \cdot \vec{m}_2 = 0$, hen the direction vectors are perpendicular and hence the lines are perpendicular.

   (c) The two direction vectors are $\vec{m}_1 = [6, -4]$ and $\vec{m}_2 = [-3, 5]$.
   Since $\vec{m}_2$ is not a scalar multiple of $\vec{m}_1$, the lines are not parallel. Also, $\vec{m}_1 \cdot \vec{m}_2 \neq 0$. Therefore the lines are not perpendicular. Hence the lines are neither parallel nor perpendicular.

5. Recall that $|\vec{z}|^2 = \vec{z} \cdot \vec{z}$. Prove that if $\vec{u} \cdot \vec{v} = 0$, then $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$.

   Using the dot product definition of magnitude and the distributive property of the dot product:

$$\begin{aligned} |\vec{u} + \vec{v}|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\ &= |\vec{u}|^2 + 2(\vec{u} \cdot \vec{v}) + |\vec{v}|^2 && \text{since } \vec{u} \cdot \vec{v} = 0 \\ &= |\vec{u}|^2 + 0 + |\vec{v}|^2 \\ &= |\vec{u}|^2 + |\vec{v}|^2 \end{aligned}$$

6. Prove **(1)** $\vec{u} \cdot \vec{u} \geq 0$ and **(2)** $\vec{u} \cdot \vec{u} = 0$ only if $\vec{u} = \vec{0}$.

   (1) By the dot product definition of magnitude $\vec{u} \cdot \vec{u} = \sqrt{(u_1)^2 + (u_2)^2}$. This is the sum of squared terms, which will always be non-negative. Therefore, $\vec{u} \cdot \vec{u} \geq 0$.

   Bad proof of (2): It will only equal 0 if $\vec{u} = \vec{0}$. That is, if $u_1 = u_2 = 0$ then $\vec{u} \cdot \vec{u} = 0$. This doesn't prove anything because we started with the end result.

   Better proof of (2): Starting with the dot product definition of magnitude and the fact we are given $\vec{u} \cdot \vec{u} = 0$.

$$\begin{aligned}
\vec{u} \cdot \vec{u} = 0 &= \sqrt{(u_1)^2 + (u_2)^2} \\
0^2 &= (u_1)^2 + (u_2)^2 && \text{square both sides} \\
-(u_1)^2 &= (u_2)^2
\end{aligned}$$

   Notice that $(u_1)^2$ and $(u_2)^2$ are both non-negative. Then notice that the left hand side is negative. The only way to have one side of the equation be negative, the other side be positive, AND have equality is to have $u_1 = u_2 = 0$.

7. Given that $|\vec{u} \cdot \vec{v}| \leq |\vec{u}|\ |\vec{v}|$, use the properties of dot products to prove

$$|\vec{u} + \vec{v}|^2 \leq (|\vec{u}| + |\vec{v}|)^2$$

   This property is called the Minkowski Inequality and can be simplified by taking the square root of both sides.

   Using the dot product definition of the dot product:

$$\begin{aligned}
|\vec{u} + \vec{v}|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\
&= \vec{u} \cdot \vec{u} + 2(\vec{u} \cdot \vec{v}) + \vec{v} \cdot \vec{v} \\
&= |\vec{u}|^2 + 2(\vec{u} \cdot \vec{v}) + |\vec{v}|^2 \\
&\leq |\vec{u}|^2 + 2(|\vec{u}|\ |\vec{v}|) + |\vec{v}|^2 && \text{since } |\vec{u} \cdot \vec{v}| \leq |\vec{u}|\ |\vec{v}| \\
&\leq (|\vec{u}| + |\vec{v}|)^2
\end{aligned}$$

   as required.