Math Circles: 2019 Intermediate Contest Preparation (Geometry)

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Welcome!

Welcome (back) to Math Circles!

This week and next week, we will be doing problem solving with the intention of preparing you for upcoming math contests.

1. Multiple Choice Contests: Pascal, Cayley, Fermat (grades 9, 10, and 11)
2. Written Contests: Fryer, Galois, Hypatia (grades 9, 10, and 11)

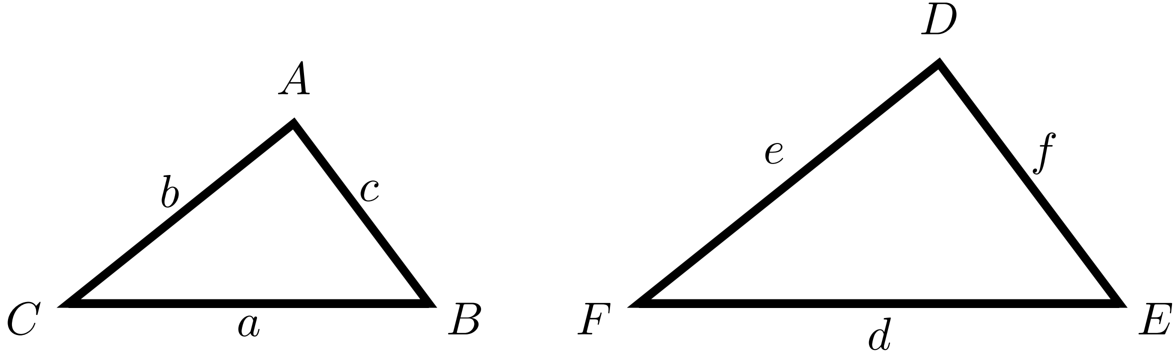
Today's Topic: Geometry Problems

Math contests tend to have lots of geometry problems on them. There are a few reasons for this:

1. There isn't much background required.
2. Geometry is historically important in math.
3. Geometry problems cause you to think in a very logical way.
4. They are fun!

Two triangles are *similar* if they have the same three angles. You can think of similar triangles as being "scaled" versions of each other.

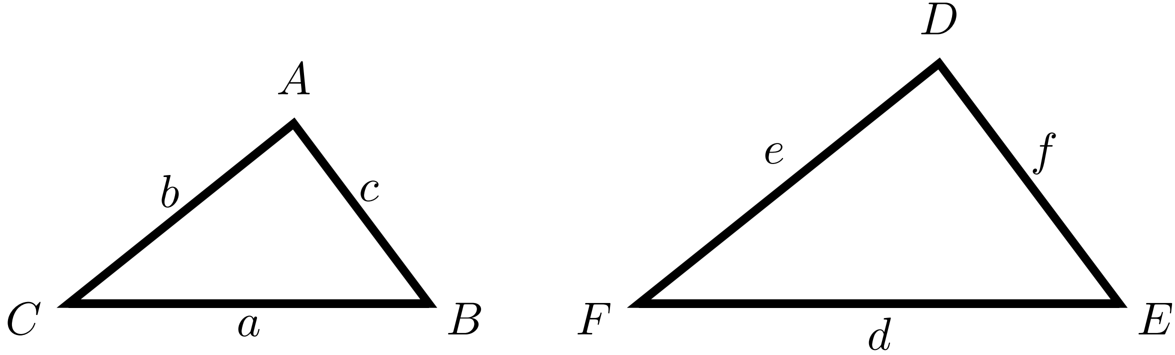
The important thing about similar triangles: **The ratios of "corresponding" sides are equal**.



$$\frac{d}{a} = \frac{e}{b} = \frac{f}{c}$$

How are the areas of similar triangles related?

Suppose $\triangle ABC$ is similar to $\triangle DEF$. Label the sides as follows: $AB = c$, $AC = b$, $BC = a$, $DE = f$, $DF = e$, and $EF = d$.



The triangles are similar, so there is some number k satisfying

$$\frac{d}{a} = \frac{e}{b} = \frac{f}{c} = k$$

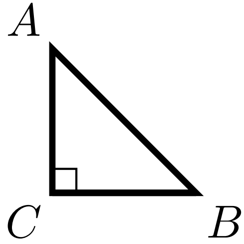
Fact

$$\frac{Area(\triangle DEF)}{Area(\triangle ABC)} = k^2$$

Useful facts about special triangles

Suppose $\triangle ABC$ has

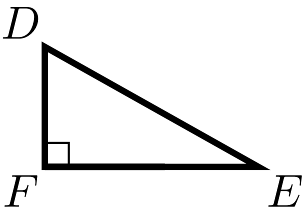
$$\begin{aligned} \angle ACB &= 90^\circ \\ \angle CAB &= 45^\circ \\ \angle CBA &= 45^\circ \end{aligned}$$



$$\boxed{\text{Then } \frac{AB}{AC} = \sqrt{2}}$$

Suppose $\triangle DEF$ has

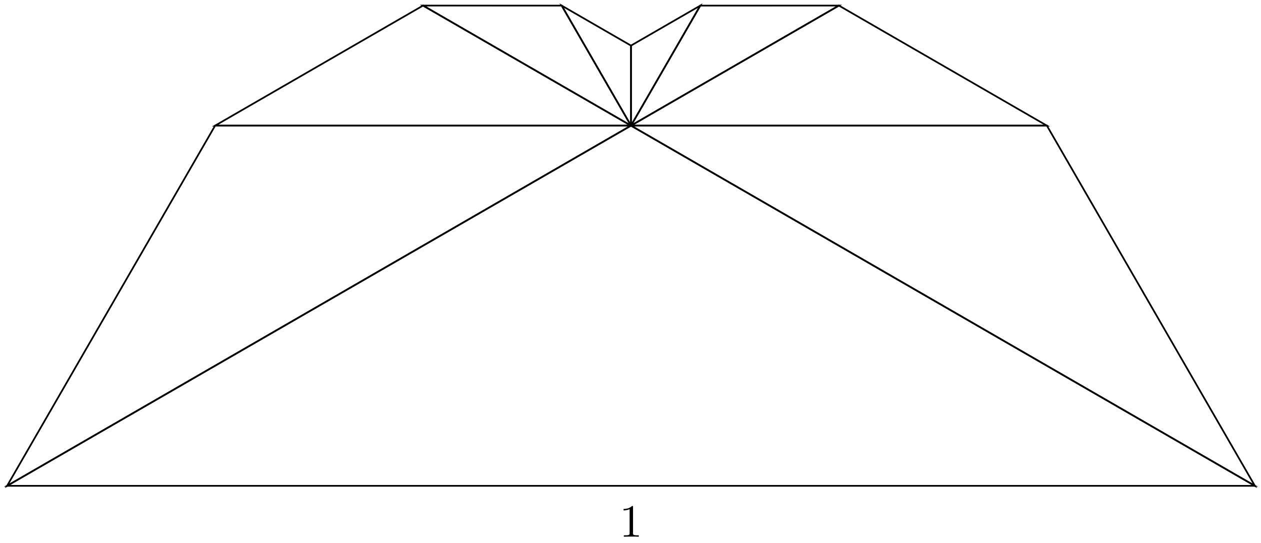
$$\begin{aligned} \angle DFE &= 90^\circ \\ \angle FED &= 30^\circ \\ \angle FDE &= 60^\circ \end{aligned}$$



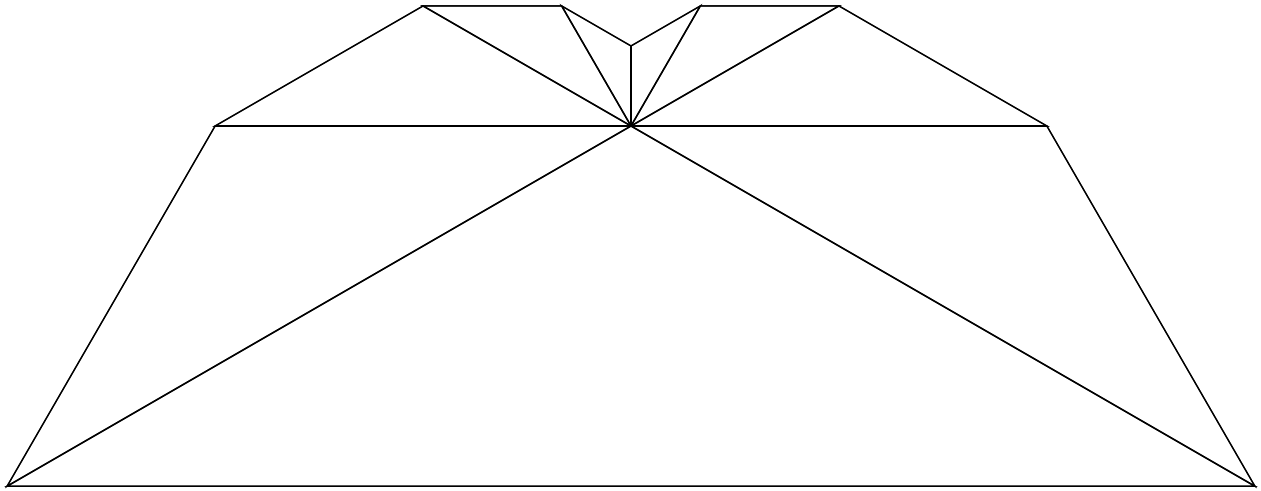
$$\boxed{\text{Then } \frac{DF}{DE} = \frac{1}{2} \text{ and } \frac{EF}{ED} = \frac{\sqrt{3}}{2}}$$

A Problem

The base of the figure below has length 1 and all triangles are isosceles and similar.

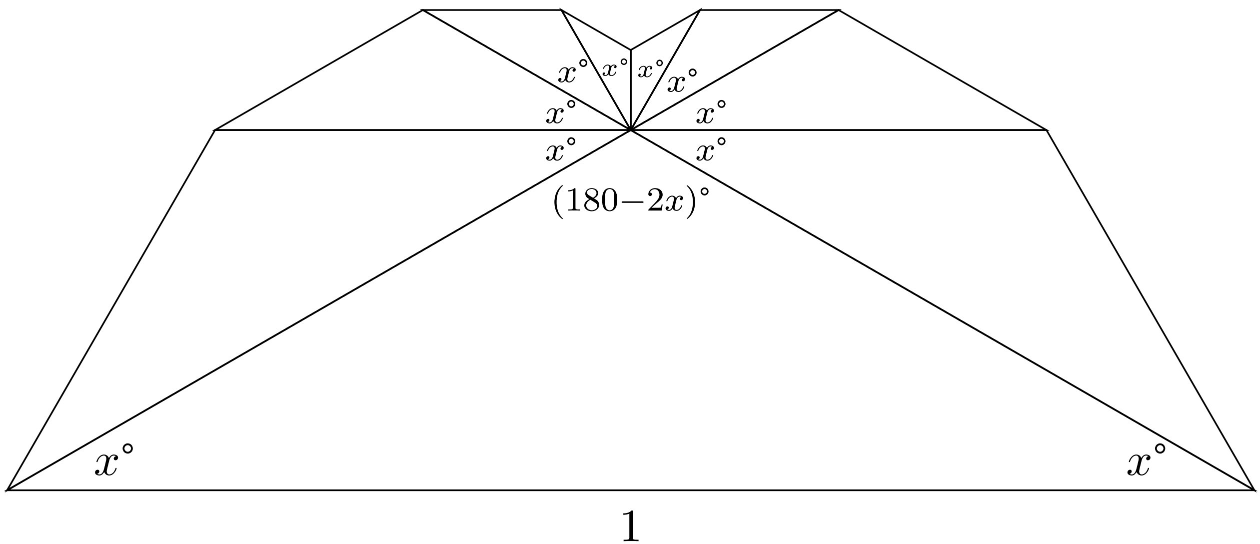


A Problem



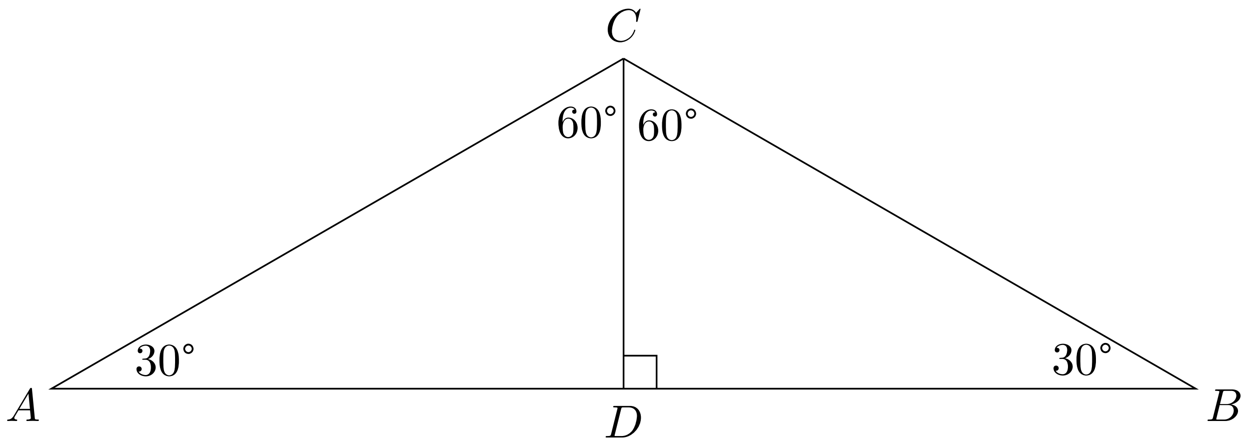
1. Find all angles.
2. Find the area of the largest triangle. [The triangle with base 1]
3. Find the total area of the figure.

Using similarity, the fact that the triangles are isosceles, and that the angles in a triangle add up to 180°, we can label the diagram as follows:



Looking at the angles around the centre of the figure, we deduce that $180 - 2x + 8x = 360$ or $6x = 180$, so $x = 30$. This means the obtuse angles are all equal to $180^\circ - 2 \times 30^\circ = 120^\circ$

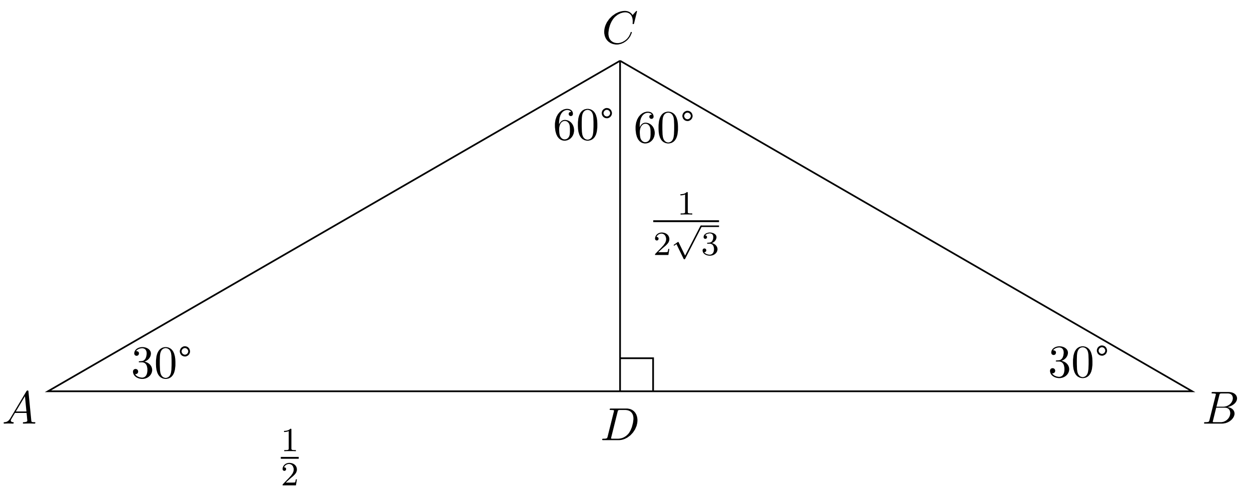
Now for the area of the large triangle.
Label the vertices A, B, and C, and drop a perpendicular from point C to meet AB at D.



$\triangle ADC$ is a 30°-60°-90° triangle, and $AD = \frac{1}{2}AB = \frac{1}{2}(1) = \frac{1}{2}$.

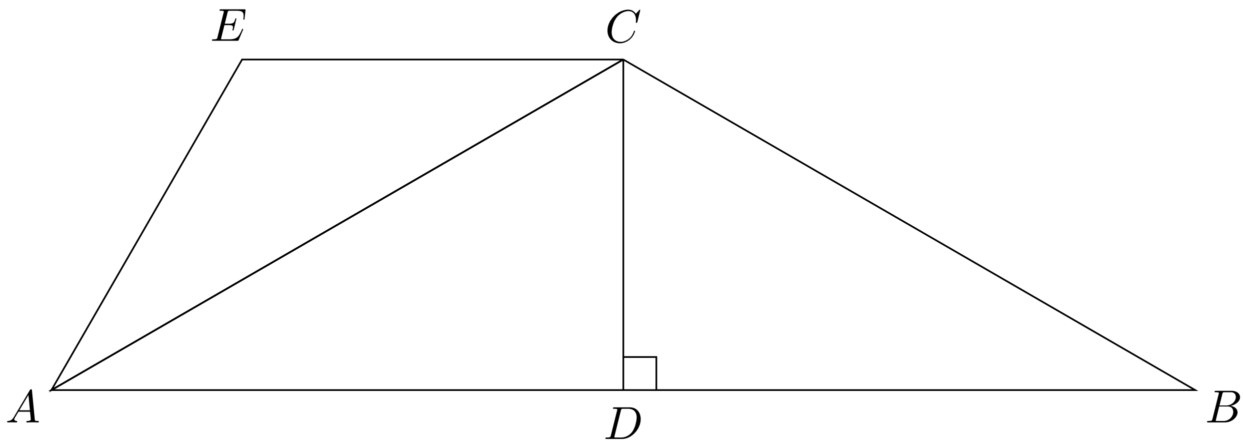
From our fact about 30°-60°-90° triangles, we have that $\dfrac{AD}{AC} = \dfrac{\sqrt{3}}{2}$ and $\dfrac{CD}{AC} = \dfrac{1}{2}$. Rearranging, we have

$$CD = \frac{AC}{2} \quad \text{and} \quad AC = \frac{2AD}{\sqrt{3}}, \quad \text{so} \quad CD = \frac{2AD}{2\sqrt{3}} = \frac{1}{2\sqrt{3}}$$



This means Area($\triangle ABC$) $= \frac{1}{2}AB \cdot CD = \frac{1}{2}(1)\frac{1}{2\sqrt{3}} = \frac{1}{4\sqrt{3}}$.

Let's find the area of one of the second-smallest triangles:



We know that $\dfrac{AD}{AC} = \dfrac{\sqrt{3}}{2}$, so $\dfrac{AB}{AC} = \dfrac{2AD}{AC} = 2\dfrac{\sqrt{3}}{2} = \sqrt{3}$.

Taking reciprocals, we have

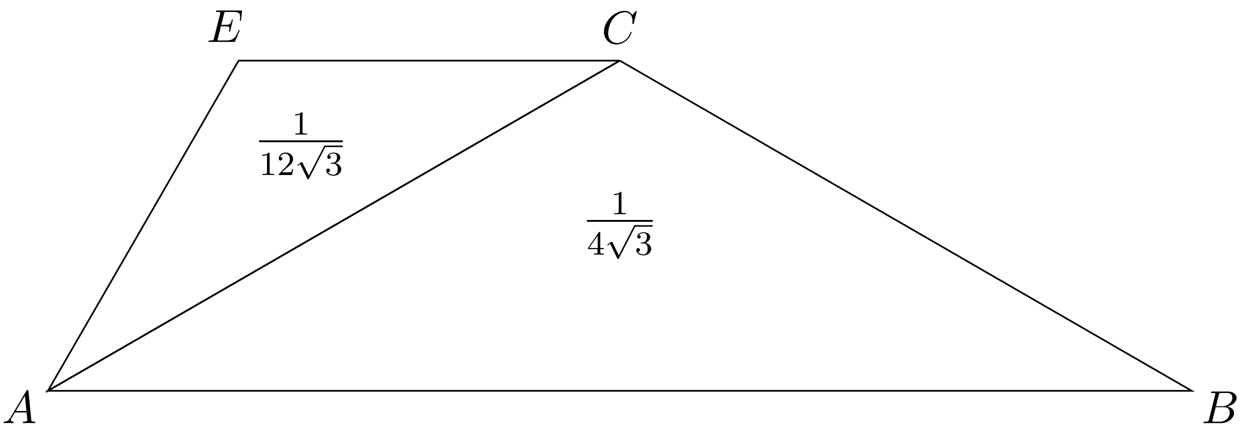
$$\frac{AC}{AB} = \frac{1}{\sqrt{3}}$$

Triangles $\triangle ACE$ and $\triangle ABC$ are similar and their bases have ratio $\frac{1}{\sqrt{3}}$. By our second fact,

$$\frac{\text{Area}(\triangle ACE)}{\text{Area}(\triangle ABC)} = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

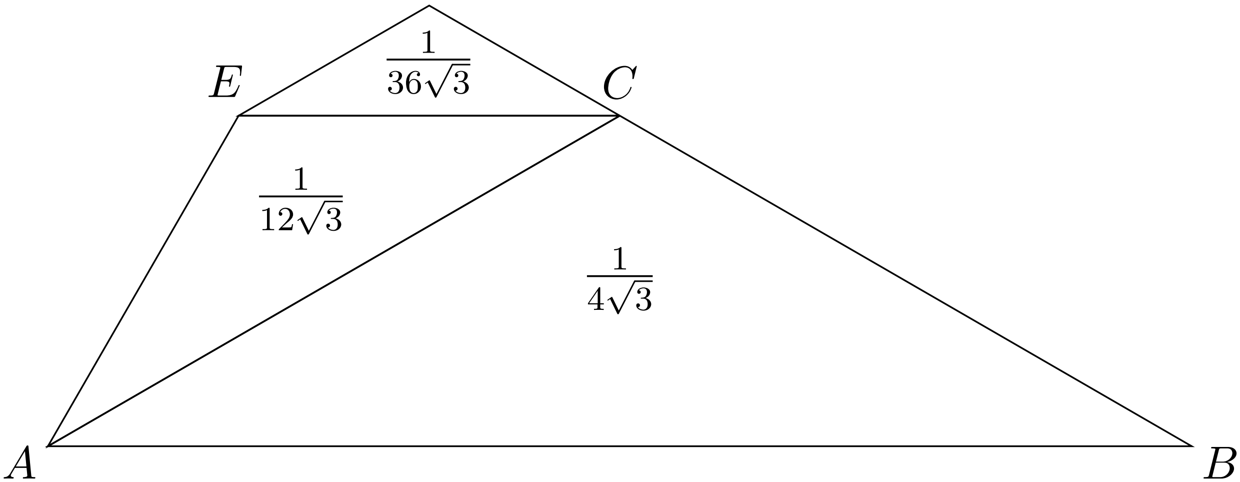
so

$$\text{Area}(\triangle ACE) = \frac{1}{3}\text{Area}(\triangle ABC) = \frac{1}{3}\frac{1}{4\sqrt{3}} = \frac{1}{12\sqrt{3}}$$



Continuing in this way, the area of the next smallest is

$$\frac{1}{3}\left(\frac{1}{12\sqrt{3}}\right) = \frac{1}{36\sqrt{3}}$$



Finally, the total area!

There areas of the "next" two triangles are $\frac{1}{108\sqrt{3}}$ and $\frac{1}{324\sqrt{3}}$. There is one large triangle, and two of each of the smaller triangles. There area of the figure is:

$$
\begin{aligned}
& \frac{1}{4\sqrt{3}} + 2\frac{1}{12\sqrt{3}} + 2\frac{1}{36\sqrt{3}} + 2\frac{1}{108\sqrt{3}} + 2\frac{1}{324\sqrt{3}} \\
= \quad & \frac{1}{4\sqrt{3}}\left(1 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81}\right) \\
= \quad & \frac{1}{4\sqrt{3}}\left(\frac{81 + 54 + 18 + 6 + 1}{81}\right) \\
= \quad & \frac{1}{4\sqrt{3}}\frac{160}{81} \\
= \quad & \frac{160}{324\sqrt{3}}
\end{aligned}
$$

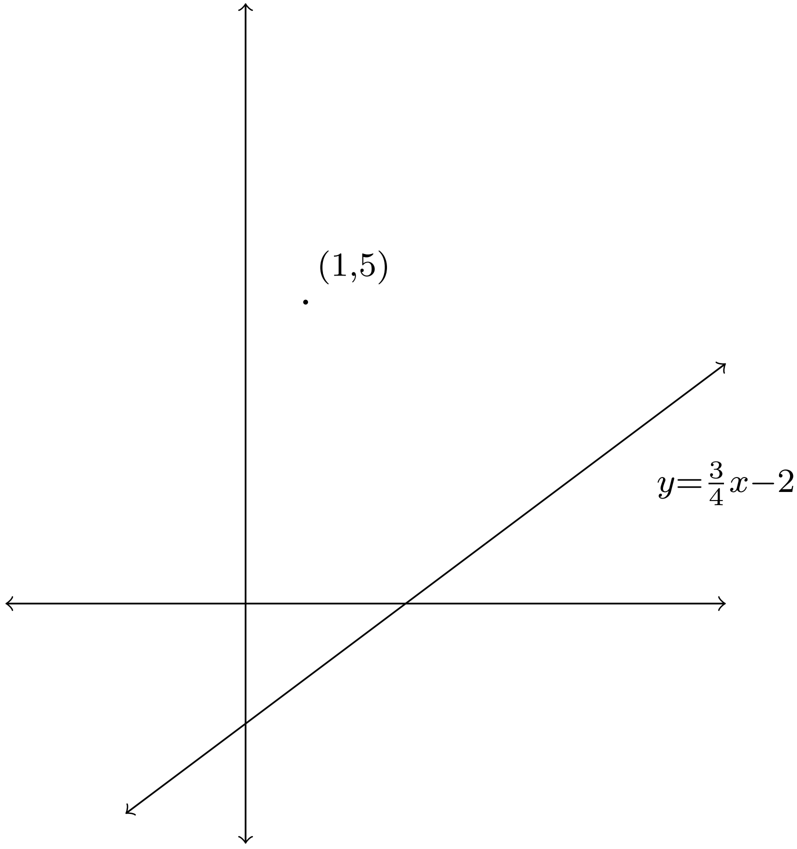
Why is the value $\frac{160}{324}$ so close to $\frac{1}{2}$?

Another Problem

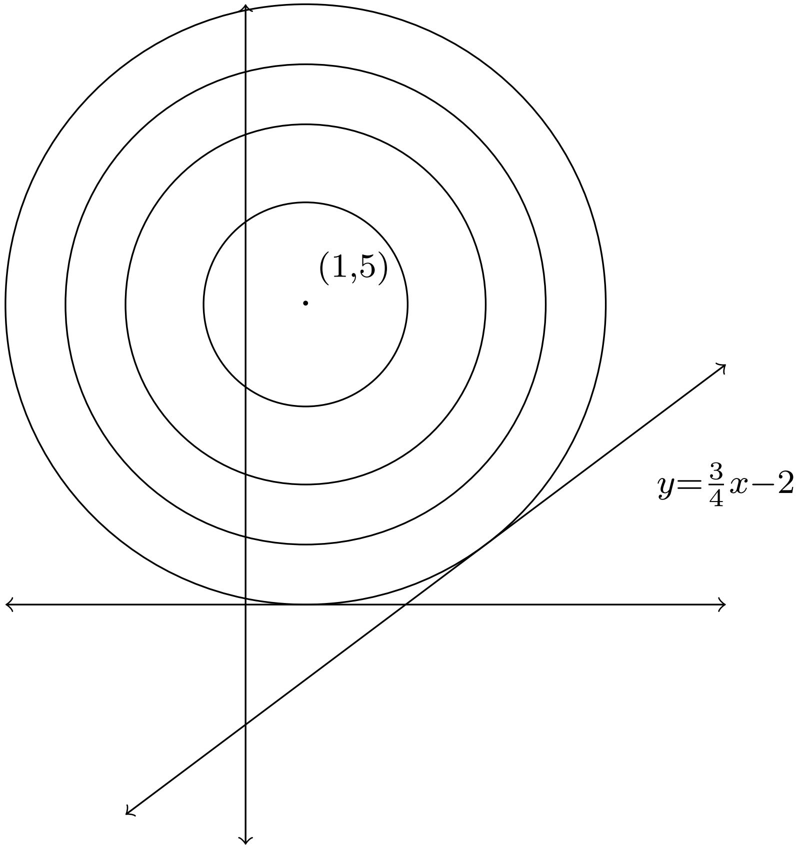
The next problem involves the equations of lines. Some grade 9 students may not have seen this quite yet. The problem was posed during the Math Circles session on February 13, but we did not go through the solution, which is included in these slides.

Another Problem

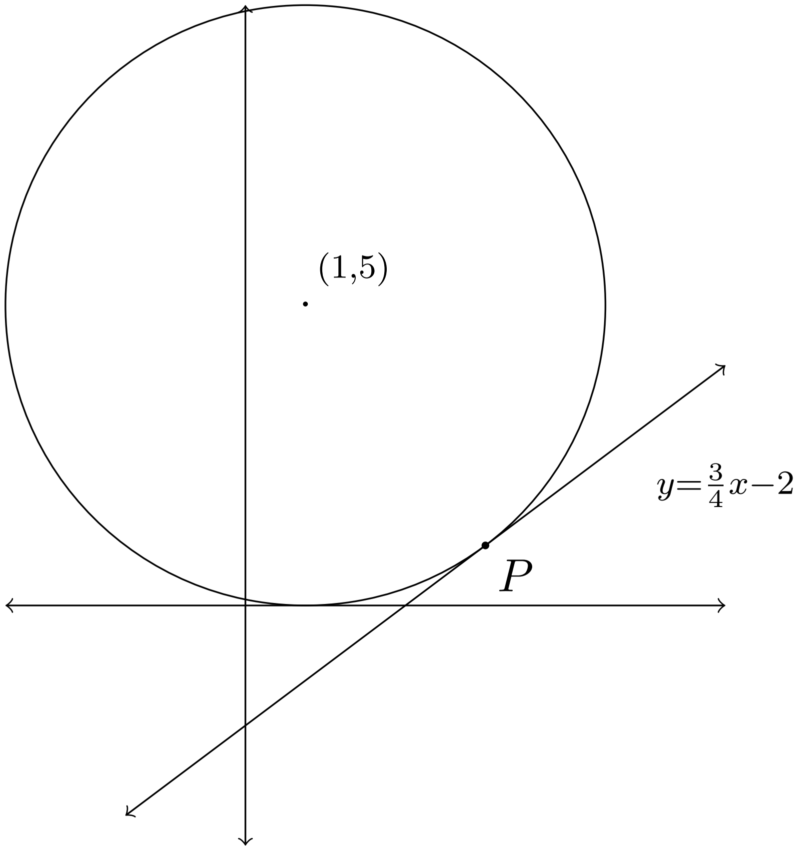
Find the point on $y = \frac{3}{4}x - 2$ which is closest to the point $(1, 5)$.



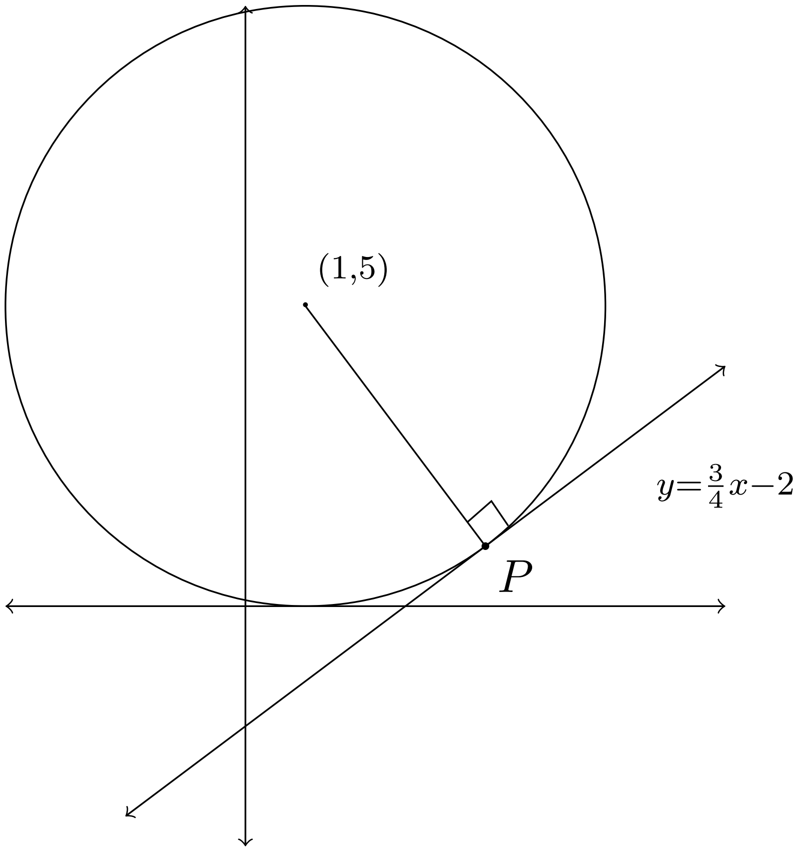
Let's draw circles centred at $(1, 5)$:



The "first" circle to touch the line should touch it at the closest point, which we will call P:



By a very important fact about circles, the radius through the *point of tangency* is perpendicular to the *tangent* [This may be a good time to look up some definitions!]:



Let's find P!

- Suppose the line defined by $(1,5)$ and P has equation $y = mx + b$.
- $y = mx + b$ is perpendicular to $y = \frac{3}{4}x - 2$.
- **The slopes of perpendicular lines are negative reciprocals of eachother**.
- Therefore, $m = -\frac{4}{3}$.
- Using that $m = -\frac{4}{3}$ and that $(1,5)$ is on the line, we have $5 = -\frac{4}{3}(1) + b$ so $b = \frac{19}{3}$.
- The point P is the intersection of

$$y = \frac{3}{4}x - 2 \quad \text{and} \quad y = -\frac{4}{3}x + \frac{19}{3}$$

Let's find P!

Setting $\frac{3}{4}x - 2$ and $-\frac{4}{3}x + \frac{19}{3}$ equal, we get

$$\begin{aligned}
\frac{3}{4}x - 2 &= -\frac{4}{3}x + \frac{19}{3} \\
\left(\frac{3}{4} + \frac{4}{3}\right)x &= \frac{6}{3} + \frac{19}{3} \\
\frac{25}{12}x &= \frac{25}{3} \\
x &= 4
\end{aligned}$$

Finally, plugging this into $y = \frac{3}{4}x - 2$, we get $y = \frac{3}{4}(4) - 2 = 3 - 2 = 1$.

$$\text{Therefore, } \; P = (4, 1)$$