Math Circles - Lesson 1
Introduction to Sequences and Series

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(I) On the first day, Becky has 8 hedgehogs on her farm. She notices that 3 new hedgehogs appear each day thereafter.

(a) How many hedgehogs will Becky have on day 7?

(b) How many hedgehogs will Becky have on day 365?

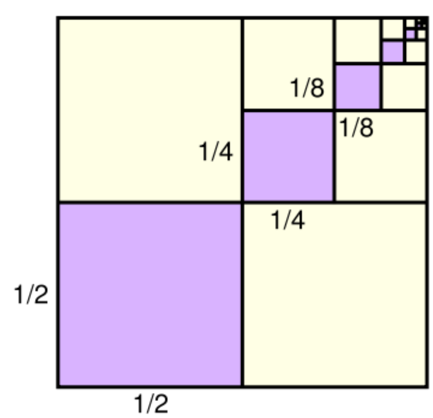
(c) Every day, Becky feeds each of her hedgehogs a pancake. How many pancakes do the hedgehogs eat in the first 31 days?

(II) Each week your grandma doubles the number of raisins in her cookies. In the first week, she puts in just 1 raisin.

How many raisins are used in week 20?

(III) You and a friend share a square pizza. Your share of the pizza is given by the purple squares in the diagram. Each square's side length is half that of the next largest square.

How much pizza do you get if an infinite number of squares are cut out?



1 Sequences

A **sequence** is an ordered list of objects. The objects are the **terms** of the sequence.

We'll be interested in sequences of numbers, but sequences can be made from anything!

Example. *Find the next term in each sequence.*

(a) $-1,\ 1,\ 3,\ 5,\ 7,$ ______ $, \ldots$

(b) $2,\ 6,\ 18,\ 54,$ ______ $, \ldots$

(c) $0,\ 1,\ 2,\ 6,\ 16,\ 44,\ 120,\ 328,$ ______ $, \ldots$

(d) 

When defining a sequence, the notation a_n is often used to denote the sequence's n^{th} term. We write $\{a_n\}_{n=1}^{\infty}$ to represent the sequence as a whole.

Example. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ given by $1, 3, 5, 7, 9, \ldots$

Fill in the blanks.

$a_1 = 1,$ $a_2 = 3,$ $a_3 =$ ______, $a_4 =$ ______, $a_n =$ ____________.

Example. Let's define a sequence $\{a_n\}_{n=1}^{\infty}$ by setting $a_n = n^2 + 3$.

We can write out the terms in this sequence by *plugging in* different values of n.

The first 6 terms in this sequence are 4, 7, ______, ______, ______, ______.

The 100^{th} term is ____________.

One can associate a sequence to each of the three word problems from the start.

(I) The sequence representing the number of hedgehogs each day is

$$\underline{\quad 8, 11, 14, 17, 20, \ldots \quad}.$$

(II) The sequence representing the number of raisins in grandma's cookies each week is

$$\underline{\qquad\qquad\qquad\qquad\qquad}.$$

(III) The sequence representing the area of each square is

$$\underline{\qquad\qquad\qquad\qquad\qquad}.$$

If a sequence is totally random, there is very little we can say about it. Instead, we will be discussing sequences that behave according to predictable patterns. Do you notice any patters in the sequences above?

2 Arithmetic Sequences

A sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is called **arithmetic** if ______________________________

____.

Example. The sequence $1, 3, 5, 7, \ldots$ is arithmetic. The common difference is _______.

Which sequence from our three word problems is arithmetic?

What is the common difference?

Arithmetic sequences are nice and predictable. So nice, in fact, that we can easily come up with a formula for the n^{th} term in the sequence.

The n^{th} Term of an Arithmetic Sequence

If a_1 is the first term of an arithmetic sequence and d is the common difference, then

$a_2 =$ ____________, $a_3 =$ ____________, $a_4 =$ ____________, etc.

In general, its n^{th} term is $a_n =$ ________________________.

Example. Let's go back to example (I). Maybe now we can help Becky with her hedgehogs!

Becky's hedgehogs can be represented with an arithmetic sequence $\{a_n\}_{n=1}^{\infty}$. The first term is $a_1 = 8$ and the difference is $d = 3$.

(a) How many hedgehogs will Becky have on day 7?

(b) How many hedgehogs will Becky have on day 365?

What about part (c)? In order to determine number of pancakes Becky will need for the first 31 days, we must compute

______________________________.

Aha! We need to know the sum of the first n terms in an arithmetic sequence. You may have seen this for the arithmetic sequence $1, 2, 3, 4, \ldots$

Here, we have

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

The Sum of an Arithmetic Sequence

If a_1 is the first term of an arithmetic sequence and d is the common difference, then

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_n = & \quad a_1 \\ & + \ (a_1 + d) \\ & + \ (a_1 + 2d) \\ & + \ (a_1 + 3d) \\ & \quad \vdots \\ & + \ (a_1 + (n-1)d) \\ \hline \end{aligned}$$

So, the sum of the first n terms is

$$a_1 + a_2 + a_3 + \cdots + a_n = \underline{\hspace{6cm}}.$$

We are now ready to help Becky! Determine the total number of pancakes eaten in the first 31 days?

3 Geometric Sequences

A sequence of numbers $\{a_n\}_{n=1}^{\infty}$ is called **geometric** if ______________________________

____.

Example. The sequence $2, 6, 18, 54, \ldots$ is geometric. The common ratio is ______.

Which sequences from our three word problems are geometric?

What are the common ratios?

The n^{th} Term of Geometric Sequence

If a_1 is the first term of a geometric sequence and r is the common ratio, then

$a_2 =$ __________, $a_3 =$ __________, $a_4 =$ __________, etc.

In general, its n^{th} term is $a_n =$ __________.

Example. Let's see if we can use our understanding of geometric sequences to answer to answer problem (II).

The number of raisins in grandma's cookies can be expressed as a geometric sequence.

The first term is $a_1 =$ __________ and the common ratio is $r =$ __________.

In week 20, grandma will use _________________________ raisins.

4 Recursive Sequences

Let's define a sequence $\{a_n\}_{n=1}^{\infty}$ in a weird way! I'm going to tell you the first two terms, and then give you a "recipe" for making the terms that come next.

Let's say the first two terms are $a_1 = 1$ and $a_2 = 1$.

To get additional terms, use the rule $a_n = a_{n-1} + a_{n-2}$.

What are the next 4 terms?

$$
\begin{array}{ccccccccccc}
a_3 & = & a_2 & + & a_1 & = & 1 & + & 1 & = & 2 \\
a_4 & = & a_3 & + & a_2 & = & \_\_\_\_ & + & \_\_\_\_ & = & \_\_\_\_ \\
a_5 & = & \_\_\_\_ & + & \_\_\_\_ & = & \_\_\_\_ & + & \_\_\_\_ & = & \_\_\_\_ \\
a_6 & = & \_\_\_\_ & + & \_\_\_\_ & = & \_\_\_\_ & + & \_\_\_\_ & = & \_\_\_\_
\end{array}
$$

This sequence is super famous! It's called the ____ and can be observed throughout the physical world.

A sequence is called **recursive** if ____

____.

Example. The sequence 0, 1, 2, 6, 16, 44, 120, 328, ... is recursive.

When $n \geq 3$, can you find a formula for a_n using the previous terms?

How can we write an arithmetic sequence $\{a_n\}_{n=1}^{\infty}$ with common difference d recursively?

How can we write a geometric sequence $\{a_n\}_{n=1}^{\infty}$ with common ratio r recursively?