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Grade 7/8 Math Circles

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Complex Numbers

Introduction

"There can be very little of present-day science and technology that is not dependent on complex numbers in one way or another." – Keith Devlin, Stanford

You may have already heard of complex numbers, especially if you've been a part of Math Circles before. Rest assured that while there may be some review, most of this should be new to you. Suppose someone wants to trick you using a math problem. You make a bet based on whether you can solve the math problem that your sneaky friend gives you. The problem turns out to be taking $\sqrt{-1}$. You, being much more clever than your friend, decide to turn the tables and trick your friend by defining a number i such that $i^2 = -1$. Your friend isn't too pleased, but you've won the bet. However, just how much sense does your new math make? After all, it must be rigorous (make sense and not lead to contradictions) to be good math...

A note about i

When we define i, we do not say that $i = \sqrt{-1}$; instead we say that $i^2 = -1$. "What's the point?" you may ask, "aren't they the same thing?" We say this because $\sqrt{\ }$ is precise notation – it denotes the function which always returns the positive answer to the question "what number, squared, gives me my first number?" The problem is that for negative numbers, the square root function gets messy and does not obey the nice properties that it usually does. As a result, when defining i, we want it to satisfy $i^2 = -1$, not be reliant on $\sqrt{\cdot}$.

We sometimes call i the "imaginary unit." This is for two reasons. First, we call all multiples of i (like $2i$, $-\frac{5}{4}i$, or even πi) the purely **imaginary numbers**. Second, recall that a negative number a can be written as $-1\times$ the positive version of that number. When taking the square root, we then get $\sqrt{-1} \times \sqrt{\text{positive version}}$. In this way, we only need to define i and not a bunch of other imaginary numbers – we can always factor it out when taking square roots.

Try it yourself

1. If $x = 2i$, find x^2.

2. If $x = -2i$, find x^2.

3. If $x^2 = -25$, what is x?

Complex numbers

Now imagine combining a real number with an imaginary number. What would you get? For example, what is $3 + 4i$?

The answer is: $3 + 4i$. We cannot combine real and imaginary numbers, so we leave them together and call them **complex numbers**. A complex number has the form $z = a + bi$, where a and b are any real numbers (numbers you're familiar with like π, $\sqrt{2}$, -3.1, 0, and 12). We call a the real part of z and b the imaginary part. Complex numbers are invaluable in their use in electrical engineering and quantum mechanics.

Adding and subtracting complex numbers

We can do arithmetic with complex numbers just like real numbers. To add two complex numbers, simply add the real parts and imaginary parts. For any two complex numbers $z = a + bi$ and $w = c + di$, $z + w$ is given by

$$\begin{aligned} z + w &= (a + bi) + (c + di) \\ &= a + bi + c + di \\ &= (a + c) + (b + d)i. \end{aligned}$$

The same goes for subtraction:

$$\begin{aligned} z - w &= (a + bi) - (c + di) \\ &= a + bi - c - di \\ &= (a - c) + (b - d)i. \end{aligned}$$

Let's try an example.

$$\begin{aligned} (4 + 2i) + (1 + i) &= 4 + 2i + 1 + i \\ &= (4 + 1) + (2 + 1)i \\ &= 5 + 3i. \end{aligned}$$

Try it yourself

4. $(1 + i) + (2i)$

5. $(10 + 7i) - (6 + 3i)$

Multiplying complex numbers

Multiplying complex numbers can be done using "**FOIL**" – that is, **f**irsts, **o**utsides, **i**nsides, **l**asts. For any two complex numbers, $z = a + bi$ and $w = c + di$,

$$\begin{aligned} z \times w &= (a + bi) \times (c + di) \\ &= \text{Firsts} + \text{outsides} + \text{insides} + \text{lasts} \\ &= ac + adi + bci + bd(i)^2 = ac - bd + (ad + bc)i \end{aligned}$$

Let's try an example.

$$\begin{aligned} (2 + 3i) \times (1 - 4i) &= (2 \times 1) + (2 \times (-4i)) + (1 \times 3i) + (3i \times (-4i)) \\ &= 2 - 8i + 3i - 12(i)^2 \\ &= 14 - 5i \end{aligned}$$

Try it yourself

6. $(1 + i) \times (2i)$

7. $(10 + 7i) \times (6 + 3i)$

8. $(4 + 3i) \times (4 - 3i)$

9. $(a + bi) \times (a - bi)$

Dividing complex numbers looks strange at first, but there's a simple trick that will help us do it. To understand the trick, we first need to introduce what's called...

The complex conjugate

Those words sound scary, but they actually correspond to a simple concept. For any complex number $z = a+bi$, the complex conjugate of z, written as $\overline{z}$, is $\overline{z} = a-bi$. You may recognize $z \times \overline{z}$ as $a^2 + b^2$ from example 9, which we'll get to later. The complex conjugate satisfies the following properties. For any two complex numbers z and w,

1. $\overline{(z+w)} = \overline{z} + \overline{w}$
2. $\overline{(z \times w)} = \overline{z} \times \overline{w}$

Try it yourself

10. $\overline{(1+i)}$

11. $\overline{(10+7i) \times (6+3i)}$

12. $\overline{1} + \overline{i}$

13. $\overline{(10+7i)} \times \overline{(6+3i)}$

Dividing complex numbers

Coming back to division, the complex conjugate is used in division to **realize the denominator**. Why do we want to do this? When you think of division, does it make sense to try and divide something into i pieces? Not really. That's why, for any two complex numbers, $z = a+bi$ and $w = c+di$, $\frac{z}{w}$ is given by

$$\begin{aligned}\frac{z}{w} = \frac{z}{w} \times \frac{\overline{w}}{\overline{w}} &= \frac{a+bi}{c+di} \times \frac{\overline{c+di}}{\overline{c+di}} \\ &= \frac{(a+bi) \times (c-di)}{(c+di) \times (c-di)} \\ &= \frac{(a+bi) \times (c-di)}{c^2+d^2}\end{aligned}$$

Try it yourself

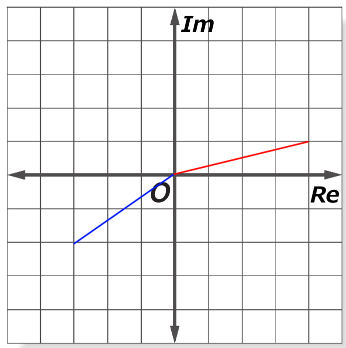
14. $\frac{2+i}{i}$

15. $\frac{-7+3i}{4+5i}$

Complex numbers as points

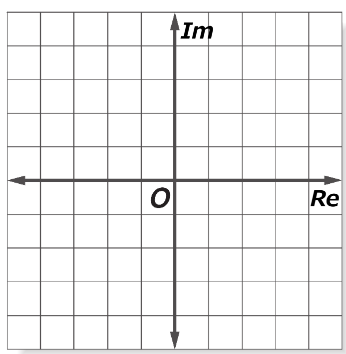
Recall the real number line. Now imagine creating a second axis consisting of the imaginary numbers, perpendicular to the reals. Then we could think of every complex number as a point in this plane. To plot a complex number $z = a + bi$, simply plot (a, b).

For example, the red line represents the complex number $4 + i$ since we plotted the point $(4, 1)$, and the blue line represents the complex number $-3 - 2i$ since we plotted the point $(-3, -2)$.



Try it yourself

Plot the complex numbers $-1 + 4i$ and $2 - 3i$.



We've got one more thing we can do with our old friend, the complex conjugate. The **modulus** of a complex number $z = a + bi$ is written as $|z|$ and is given by $|z| = \sqrt{z \times \overline{z}} = \sqrt{a^2 + b^2}$. (Note that here we mean the positive square root.) Does this look familiar? What could the modulus represent?

Try it yourself

Find the modulus of the complex numbers from the previous example.

Use in science and technology

You may be thinking to yourself, "well this is neat, but what's the point? How do we use complex numbers in real life?" Believe it or not, complex numbers are used in a variety of places, most often in things that involve *waves*. There is a very nice connection between waves and complex numbers that requires more advanced math than we have time for right now. That connection allows us to write things nicely and neatly, allowing us to do things more efficiently and giving rise to ideas that we otherwise wouldn't have come to. Some examples of things that you use in your everyday life that are the way they are because of complex numbers include: radio, television, digital music and video playback, the internet, LEDs (the lights that power our screens), the circuitry that powers our computers, and so on. The list is nearly endless. Almost anything you can think of that involves small electronic circuits, right down to the digital watch, are possible thanks to complex numbers and their use in quantum theory and electrical engineering.

Problems

1. Evaluate the following:
   (a) $(1+i)-3i$
   (b) $(5-2i)+(2+i)$
   (c) $-2(7-8i)+3(3+2i)$
   (d) $(2-i)+3(2+i)-2(1+3i)$
   (e) $(6-8i)+2\overline{(2+5i)}$
   (f) $4(3+2i)-5\overline{(1-i)}+6\overline{(-3+i)}$

2. Evaluate the following:
   (a) $(3+i)\times 2i$
   (b) $(-5+3i)\times(1+2i)$
   (c) $4(2-3i)\times 2(1-4i)$
   (d) $(-1+2i)\times 2(5+2i)\times 3(1+6i)$
   (e) $(-3-4i)\times 4\overline{(-3+i)}$
   (f) $2(4+i)\times -3\overline{(9-i)}\times\overline{(-1-i)}$

3. Evaluate the following:

(a) $(2+3i) \div i$

(b) $(3-4i) \div (1+9i)$

(c) $-(1-7i) \div 3(4+2i)$

(d) $((2-2i) \div (5+5i)) \div 6(2-3i)$

(e) $(1-2i) \div 2\overline{(4+3i)}$

(f) $\left(5(1+3i) \div 2\overline{(2-3i)}\right) \div -6\overline{(1-4i)}$
(Feel free to use a calculator)

4. Plot the following complex numbers:

(a) $2+3i$

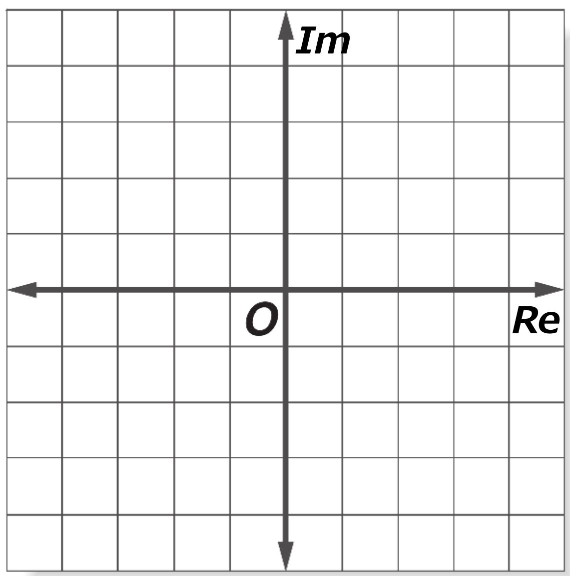
(b) $3-4i$

(c) $-1-3i$

(d) $-2+2i$

(e) $2(1-2i)-(4-3i)$

(f) $(1+3i) \times (2-i)$



5. Find the moduli of the complex numbers from problem 4.

6. Prove the following. For any two complex numbers z and w,...

(a) $\overline{(z+w)} = \overline{z} + \overline{w}$

(b) $\overline{(z \times w)} = \overline{z} \times \overline{w}$

(c) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

7. Quadratic equations:

You're probably familiar with solving what are known as **linear** equations involving x. A **quadratic** equation is one of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers. As you'll learn eventually, we can always find solutions to these kinds of problems. There's even a really nice formula that lets us find solutions rather easily. This formula is called the quadratic formula and it is: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where x is (are) the solution(s) to the equation above, and "±" stands for "plus **or** minus." In many cases there are two numbers that solve the equation, and the "±" accounts for them.

Let's try an example together. We'll solve $x^2 - 3x + 2 = 0$:

1. Identify a, b, and c: Here, $a = 1$, $b = -3$, and $c = 2$

2. Fill in the spots in the quadratic formula: $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)}$

3. Simplify:

$$
\begin{aligned}
x &= \frac{3 \pm \sqrt{9 - 8}}{2} \\
&= \frac{3 \pm 1}{2} \\
\text{Here's where we split the “} + \text{” and the “} - \text{”:} &= \frac{4}{2} = 2 \textbf{ or } \frac{2}{2} = 1.
\end{aligned}
$$

We can now check our answer by **substituting** our answers back into the original equation:

$$(1)^2 - 3(1) + 2 = 1 - 3 + 2 = 0$$

and

$$(2)^2 - 3(2) + 2 = 4 - 6 + 2 = 0.$$

Now you're equipped to try it on your own!

Using the quadratic formula, solve the following quadratics:

(a) $x^2 - 1 = 0$

(b) $x^2 + 2x + 1 = 0$

(c) $2x^2 - 6x - 8 = 0$

(d) $x^2 + 1 = 0$

(e) $x^2 - x + 4 = 0$

(f) $3x^2 + 2x - 8 = 0$