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Grade 7/8 Math Circles

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Exponentiation

Powers

When you first learned to multiply, you probably learned to think of it as repeated addition; for example, 4×3 = adding four three times (or adding three four times) $= 4+4+4 = 12$. In much the same way, we can introduce a new operation which is like repeated multiplication. For example, if we wanted to express four times itself three times, $4 \times 4 \times 4$, we could write it like this: 4^3. We call this operation ________________, and we represent it like this: $\text{base}^{\text{exponent}}$. When reading this notation aloud, we say "(base) to the power of (exponent)," or sometimes just "(base) to the (exponent)." An important thing to notice is that this is not like multiplication in the sense that *order matters*. For example, $9^2 = 81 \neq 2^9 = 512$. In general, for two numbers a and b, $a^b \neq b^a$.

Try it yourself #1

Evaluate the following:

1. 4^3
2. 2^5
3. 8^2

You may already have questions such as, "if we can exponentiate with any numbers, what about zero?" or, "what about one?" or even, "what about negative numbers?" Let's think about the answers to those questions.

Try it yourself #2

To answer some of the above questions, attempt the following:

1. 1^2
2. 1^3
3. 1^{2019}
4. 0^2
5. 0^3
6. 0^{2019}

As you can see, if 1 is our base, then we can multiply it by itself as many times as we like and we'll always get 1. The same is true of 0.

Try it yourself #3

Evaluate:

1. 2^1
2. 3^1
3. 2019^1

If one is our *exponent*, then we just get our base back—multiplying a number once gives the number we started with. Zero as an exponent is a little strange, so we'll come back to it later. Negative exponents are also a little strange, but we'll go through them now and explain them later. A number raised to a negative power is simply the *reciprocal* of the number raised to that power. The reciprocal of a number is 1 divided by that number. In symbols, where a is a number and x is a positive number, $a^{-x} = \frac{1}{a^x}$. For example, the reciprocal of 5 is $\frac{1}{5}$. In exponential notation, we can say that the reciprocal of 5 is just 5^{-1}.

Try it yourself #4

Evaluate (with a calculator if necessary) or at least simplify:

1. 2^{-3}
2. 5^{-2}
3. 10^{-4}

We can use exponentiation with almost any numbers we like, it just becomes a little more difficult to think about. For example, if you try $4.1^{3.3}$ on your calculator, you'll get roughly 105.241426756. How did the calculator do this? To understand, we have to talk a bit more about the exponential.

Roots

First, however, we are going to talk about roots. You have all seen the symbol, "$\sqrt{\text{number}}$" before. This symbol denotes the ____________ and means "what number, when I multiply it by itself (*square* it), gives me my original number?" For example, $3 \times 3 = 9$ so $3 = \sqrt{9}$. (Observe that $-3 \times -3 = 9$ as well). Notice also that we cannot take the square root of a negative number—no real number times itself is negative! We can extend the question we asked earlier—that is, make it more general—and ask instead, "what number do I have to multiply by itself n times to get my original number," where n is just a whole number. We write that question down as, "$\sqrt[n]{\text{number}} = ?$" and call the answer the n^{th} root of the number.

Try it yourself #5
Evaluate the following roots:
$\sqrt{49}$,

$\sqrt{81}$,

$\sqrt[3]{-27}$,

$\sqrt[5]{32}$.

Again, we can take almost any number to any power we like. This means that we can, for example, take $4^{\frac{1}{2}} = 2$.

Try it yourself #6
Evaluate the following with a calculator:

1. $49^{\frac{1}{2}}$
2. $81^{\frac{1}{2}}$
3. $(-27)^{\frac{1}{3}}$
4. $32^{\frac{1}{5}}$

Notice anything? This is because $\sqrt[n]{\ }$ and $\frac{1}{n}$ are actually two different ways to say the same thing! It turns out that we can write powers and roots interchangeably. This is due to the following property of exponents:

Property #1: For any real number a and any two exponents, x and y, $a^x \times a^y = a^{x+y}$.

Try it yourself #7
Evaluate the following:

1. $2^2 \times 2^2$
2. $3^2 \times 3$
3. $5^2 \times 5^2$
4. $6^1 \times 6^{-1}$

We're finally in a position to understand what fractions and zero do as exponents. To understand zero, recall that any number times its reciprocal is one, *i.e.* $a \times \frac{1}{a} = 1$. We can write this in exponential notation as $a^1 \times a^{-1} = 1$. We know the property of exponents that we discussed earlier, so we can see that $a^1 \times a^{-1} = a^{1-1} = a^0$. So a^0 should be ___. (This is true in every case except when $a = 0$. Things with zero can be weird sometimes.)

To understand roots, recall that $\frac{1}{2} + \frac{1}{2} = 1$. What if we combine this with the first exponent property? We can see that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$. Does this look familiar? It's kind of hard to see—disguised, almost—but when we ask "what number multiplied by itself gives me my starting number," this is another way to think about it. The same idea applies to third, fourth, and higher roots, for example, $2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}} = 2$.

For fractions as the base, we simply take the numerator to the given exponent, and the denominator to the given exponent. For example, $\left(\frac{5}{4}\right)^2 = \frac{5^2}{4^2} = \frac{25}{16}$.

So far, we've only seen reciprocals of whole numbers. What about other fractions? To learn about these, we'll introduce another property of exponents.

Property #2: For any non-negative base a and any two exponents, s and t, $(a^s)^t = a^{s\times t}$.

Now if we wanted to figure out what $8^{\frac{2}{3}}$ is, we can first "reverse-engineer" the exponent into $8^{2\times\frac{1}{3}}$. Now we can recognize this as $\sqrt[3]{8^2}$ and simplify: ______.

Try it yourself #8

Simplify and evaluate the following if able:

1. $(2^2)^2$
2. $(3^4)^{\frac{1}{2}}$
3. $(5^2)^{-1}$
4. $(6^1)^{-2}$

The nicest base

There is one number in particular that mathematicians like taking powers of the most. You *may* have heard of it before. It is called Euler's number after Leonhard Euler (he did not discover it, but it got named after him anyway). Euler's number is written as e, and it is approximately equal to 2.718. You may think to yourself "well that's a pretty ugly-looking number. Why would we want to take powers of that?" It turns out that e has a variety of very useful properties that make it the ideal base for exponents. The problem is that pretty much all of those properties require higher-level math that we don't have time for. We can, however, at least get a sense of where e comes from...

Try it yourself #9

Evaluate the following with your calculator. Do you notice anything?

1. $\left(1+\frac{1}{1}\right)^1$
2. $\left(1+\frac{1}{2}\right)^2$
3. $\left(1+\frac{1}{3}\right)^3$
4. $\left(1+\frac{1}{50}\right)^{50}$
5. $\left(1+\frac{1}{100}\right)^{100}$
6. $\left(1+\frac{1}{5000}\right)^{5000}$

This comes up in the problem of **compound interest**. When you put your money in the bank, you earn interest on it, meaning that the bank gives you extra money for having your money in the bank. The bank can give you money using one of two methods—either simple or compounded. Simple interest isn't important right now. Compound interest means that the amount of money the bank gives you is based on three things: (1) **how much** money you have in the bank, sometimes called the **principal**, (2) the interest **rate**, and (3) the **compounding period**. Let's go through these. How much you have is straightforward. The interest rate is a percentage and determines how much money the bank actually gives you. The amount is how much you have (including interest) × that percentage. Finally, the compounding period is how often the bank gives you money. The interest rate is divided by the compounding period when calculating, otherwise the bank would just be giving you more and more money!

Let's try an example. Suppose you have an account with \$10000 in it and a 3% interest rate, compounded daily. To calculate how much money you will have in the future, we use the following formula: $A = P\left(1 + \frac{r}{n}\right)^{nt}$. Let's break that down.

1. $A =$ the amount you're going to have,
2. $P =$ the amount you start with, also called the **principal**,
3. $r =$ the interest rate in decimal form,
4. $n =$ the number of times per year that interest is paid, and
5. $t =$ the amount of time in years.

If we want to know how much money will be in our account five years from now, we identify our variables and substitute into the formula:

1. $A =$ what we're trying to figure out
2. $P = \$10000$,
3. $r = 0.03$ since $3\% = 0.03$,
4. $n = 365$ since interest is paid daily and there are 365 days in the year, and
5. $t = 5$ years.

Substituting gives $10000\left(1 + \frac{0.03}{365}\right)^{5\times365} = \$11618.27$! We would earn \$1618.27 by doing nothing! If instead we only invested one dollar, had an interest rate of 100% that compound every second, and waited for one year, we'd get \$2.72 at the end of it. In other words, we get e by trying to calculate compound, 100% interest at very fast compounding intervals over very long times.

Try it yourself #10

If we have \$400 000.00 in a bank account that pays 5% interest compounded monthly, how much will we have after 3 years?

Scientific notation

We are often interested in using powers of ten to give an estimate of how large things are compared to other things. We call this notion **order of magnitude**. We say two things are of the same order of magnitude if we can express their sizes using the same power of ten in **scientific notation**.

What is scientific notation? It allows us to write numbers as some coefficient times a power of ten, so we can easily compare relative sizes of numbers. To write a number in scientific notation, move the decimal point from one end of the number to the other, between the first and second nonzero digits when reading from left to right. Then count how many places you moved the decimal point. That number is the power of ten that you'll use. For decimal numbers, use the negative version of that number. Then write your new decimal number times the appropriate power of ten. Let's try a couple of examples.

Example:

Convert the following to scientific notation:

1. 5 396 000

   We move the decimal point from the back to the front, between the first two nonzero digits when reading left to right – between the "5" and the "3." We have to move the decimal point six positions, so our power of ten is 10^6. Our scientific notation, then, is 5.396×10^6.

2. 0.0003174

   We move the decimal point from the front to the back, between the first two nonzero digits when reading left to right – between the "3" and the "1." We have to move the decimal point four positions, so our power of ten is 10^{-4}. Our scientific notation, then, is 3.174×10^{-4}.

Try it yourself #11

Express the following in scientific notation:

1. 3520
2. 1165800000
3. 0.00365
4. 0.000094651

Returning to orders of magnitude, we'll now see how to compare them. For example, the distance between the earth and the sun is approximately 1.5×10^8 km, while the distance between earth and Saturn is at least 1.2×10^9 km away. Because these two numbers differ in power of ten by one, we say that the distance between earth and Saturn is one order of magnitude larger than the distance between the earth and the sun.

Try it yourself #12

Which of the following are of the same order of magnitude?:

1. Number of seconds in a year and the number of people in Canada
2. Population of the world and number of stars in the solar system
3. Number of pixels in a full HD screen and the distance between the earth and the moon
4. Width of a human hair and width of a grain of sand

Enjoy the problems below!

Problems

1. Evaluate the following:
   (a) 5^2
   (b) 3^3
   (c) 4^{-1}
   (d) $9^{\frac{1}{2}}$
   (e) $64^{-\frac{1}{3}}$
   (f) $8^{-\frac{2}{3}}$

2. Use the first property of exponents to simplify:
   (a) $2^2 \times 2^3$
   (b) $3^8 \times 3^{-6}$
   (c) $\sqrt{5} \times 5^{\frac{3}{2}}$
   (d) $8^{19} \times \frac{1}{8^{20}}$
   (e) $11^2 \times \frac{1}{\sqrt[3]{11}} \div 11^{-\frac{2}{3}}$

3. Use the second property of exponents to simplify:

(a) $(2^2)^{\frac{1}{2}}$

(b) $(3^{-1})^{-3}$

(c) $\left(\left(\frac{1}{5}\right)^2\right)^{-\frac{3}{2}}$

(d) $\left(8^{\frac{1}{3}}\right)^{\sqrt[3]{27}}$

(e) $\left(\left(\frac{1}{\sqrt{11}}\right)^{\frac{1}{4}}\right)^{-\sqrt{16}}$

4. Solve for x:
   (a) $10^x - 10 = 9990$
   (b) $4^x = 64^2$
   (c) $3^6 = 27^x$
   (d) $5^{33} = 125^x$
   (e) $6^{x+2} = 216$
   (f) $8^{x-1} = 2^6$

5. Calculate the following amounts assuming compound interest:
   (a) ...after one year, 10% interest, principal of \$500, compounded monthly
   (b) ...after five years, 2% interest, principal of \$1500, compounded quarterly (every four months)
   (c) ...after ten years, 4% interest, principal of \$1 000 000, compounded annually
   (d) ...after three years, 1.8% interest rate, principal of \$5000, compounded weekly
   (e) ...after seven years, 2.7% interest rate, principal of \$2001, compounded monthly

6. Jimmy Bob gets lucky and wins 10 million dollars in the lottery. He decides to put the money in a savings account with a 1.5% interest rate that compounds monthly. How much money will Jimmy Bob have to live on if he lets his money accumulate interest for a year, then withdraws the interest money and lives on it?

7. Describe the difference in order of magnitude between the following:
   (a) Number of seconds since the approximate start of the universe ($\sim 4.3 \times 10^{16}$) and the approximate number of stars in the observable universe ($\sim 1.0 \times 10^{21}$)
   (b) "Size" of the electron[1] ($\sim 2.8 \times 10^{-15}$ m) and charge radius[2] of the proton ($\sim 8.8 \times 10^{-16}$ m)
   (c) Radius of the human red blood cell ($\sim 3.6 \times 10^{-6}$ m) and width of the finest spider silk ($\sim 3 \times 10^{-6}$ m)
   (d) Mass of an elephant ($\sim 6.0 \times 10^3$ kg) and the mass of a blue whale ($\sim 1.4 \times 10^5$ kg)

[1]Electrons are actually point particles, but it is useful to think about this regardless.
[2]That is, how far away from the proton its charge extends.

(e) The height of the tallest building in the world, the Burj Khalifa ($\sim 8.3 \times 10^2$ m), and the height of the average human ($\sim 1.7 \times 10^0$ m)

(f) The maximum depth of the ocean ($\sim 1.1 \times 10^4$ m) and the height of Mount Everest ($\sim 8.8 \times 10^3$)

8. **Challenge:** Find two distinct (non-equal) integers, x and y, such that $x^y = y^x$.