UNIVERSITY OF WATERLOO

Faculty of Mathematics

Waterloo, Ontario

Centre for Education in

Mathematics and Computing

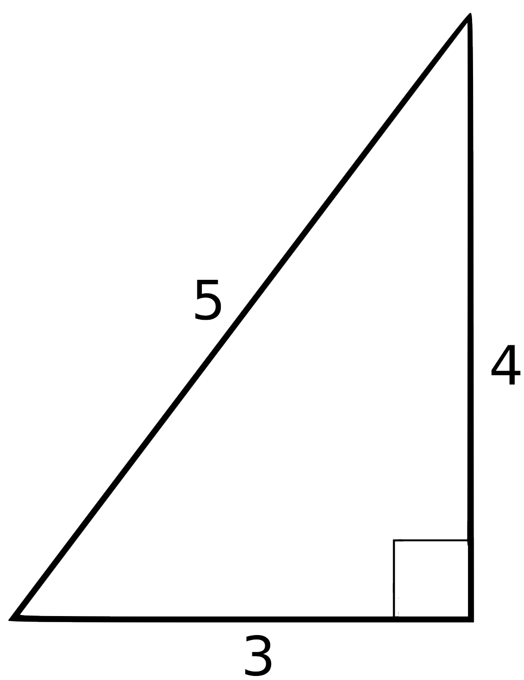
Grade 7/8 Math Circles

Winter 2019 – Feb. 26/27/28

The Pythagorean Theorem

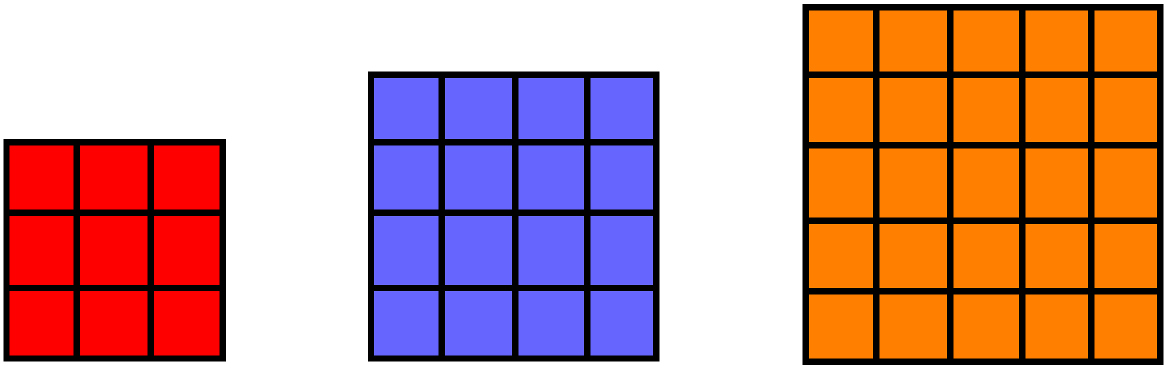
Introduction

A right triangle is any triangle with a 90° angle. Let's take a look at one!



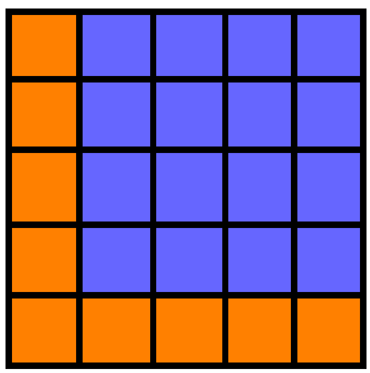
At first glance, there doesn't seem to be anything special about this triangle; but there **IS** something special about the lengths of its 3 sides. To illustrate, let's take the square of each side from the triangle above and draw some pictures!

These are the squares of each respective side of the right triangle from the page before:

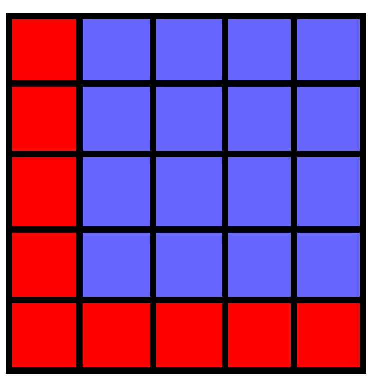


3^2 4^2 5^2

Just because we can, let's see if our two smaller squares fit into the bigger one:



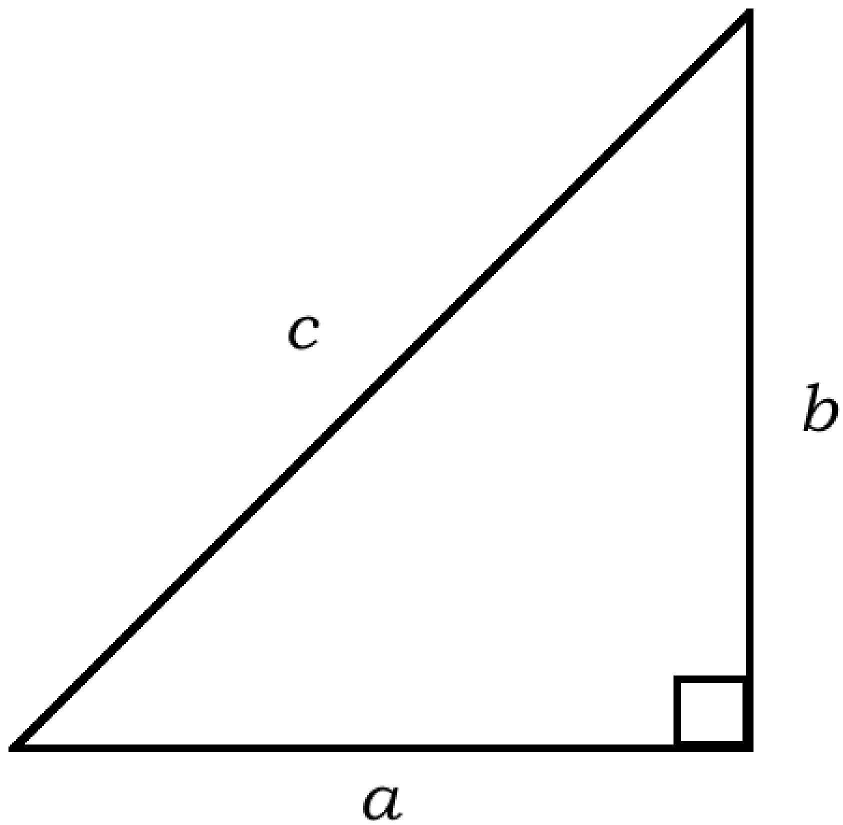
Now, it doesn't look like we could fit our 3 by 3 square in there; but notice that there are 9 orange, 1 by 1 squares left, which is the same amount of red, 1 by 1 squares that make up our 3 by 3 square! Let's break apart our red, 3 by 3 square and fit it into the rest of our 5 by 5 square above.



Well, it seems like the squares of the two smaller sides of the right triangle can be used to make up the square of the biggest side. In particular, $3^2 + 4^2 = 5^2$.

The Pythagorean Theorem

What we discovered in the example above is in fact true for any right triangle.



In particular, for any right triangle, if a and b are the lengths of the two sides that meet at the right angle ($90°$), and c is the length of the longest side, opposite of the right angle, then $a^2 + b^2 = c^2$.

Let's introduce some triangle terminology so we can make this theorem look nicer.

Triangle Terminology:

vertex - any corner of the triangle.

adjacent sides - two sides are said to be adjacent if they meet at a common vertex.

legs - the two adjacent sides of a right triangle that meet at the right ($90°$) angle.

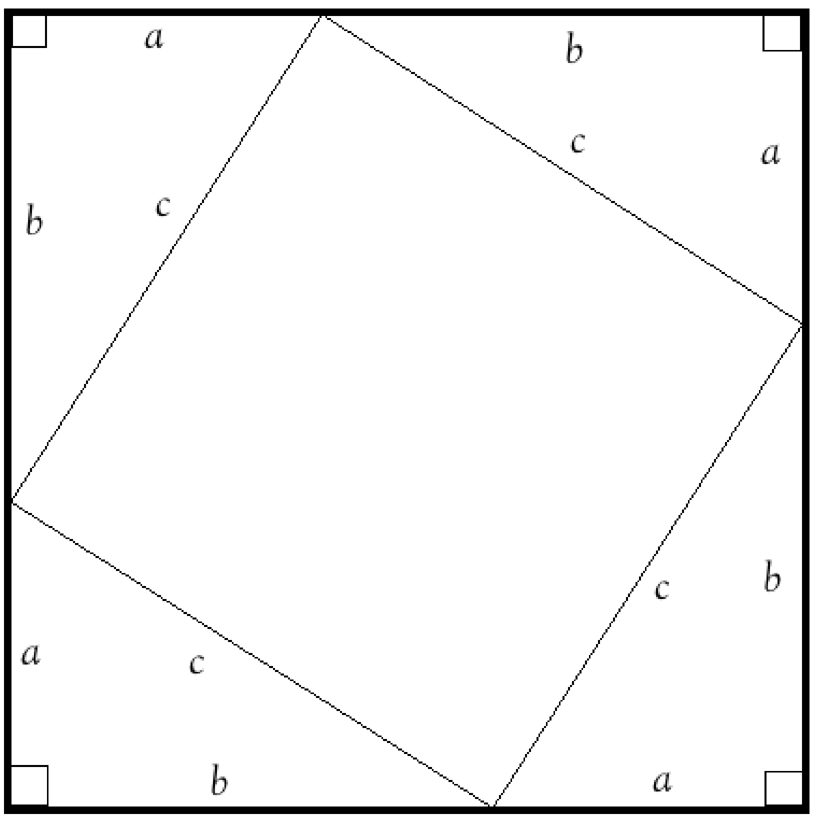
hypotenuse - the longest side of a right triangle, opposite of the right angle.

The Proof

Prove the following:

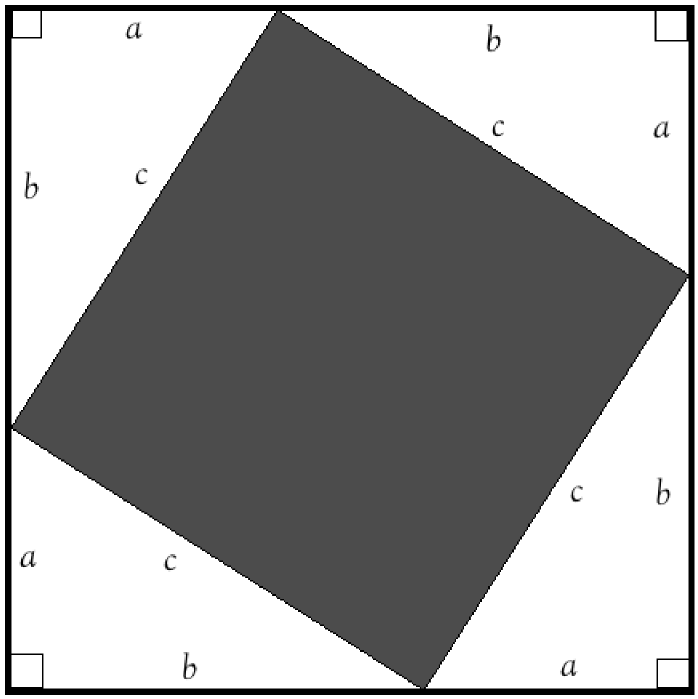
For any right triangle, if a and b are the lengths of the legs, and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.

We will prove this theorem by using pictures. Let's arrange 4 of the same right triangles to get the figure below:

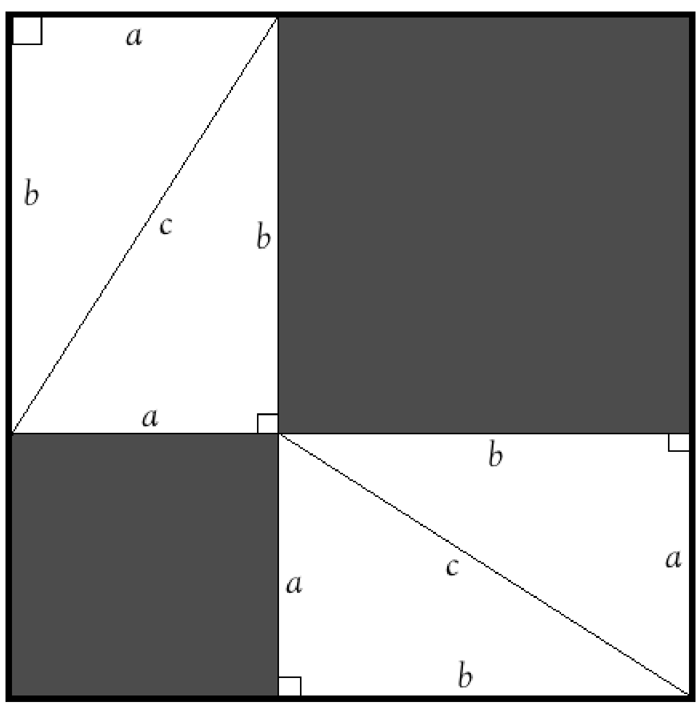


Notice that the 4 right triangles have the same adjacent sides a and b that meet at a 90° angle. So, because the corresponding sides of each triangle have equal length and meet at the same angle, they are all the same triangle (this follows from the axiom in math called "side, angle, side" or **SAS**). More importantly, this tell us that these 4 triangles have the same hypotenuse. We'll call the length of this hypotenuse c.

Well, if the 4 side lengths of the figure in the middle are all c, this tells us we have a square. The shaded region represents the area of the square in the middle of our figure. This area is equal to $c \times c$ or c^2.



Now, remember that we need to show that $a^2 + b^2 = c^2$. So, because we can, let's play around with the arrangement of the 4 right triangles in the figure above:

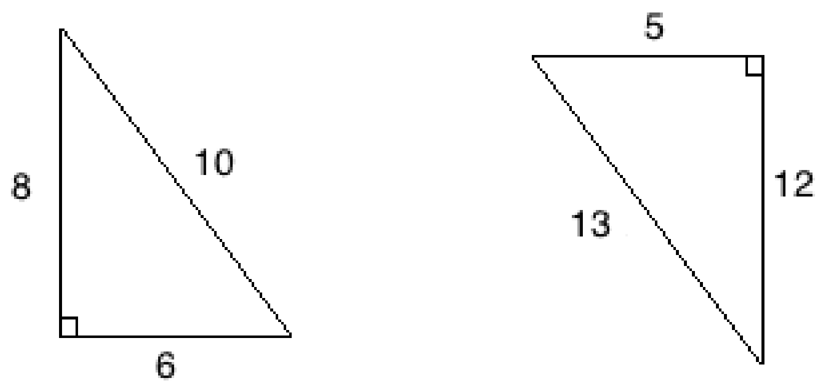


Notice that the total area of our figure did not change, because all we did was move the right triangles around. Then, since the area of the 4 right triangles has also not changed, this tells us that the area of our two shaded regions must equal the area of our original shaded region, c^2. But, the area of the smaller shaded region is just $a \times a$, or a^2, and the area of the larger shaded region is just $b \times b$, or b^2.

Therefore, we conclude that $a^2 + b^2 = c^2$. $\square$

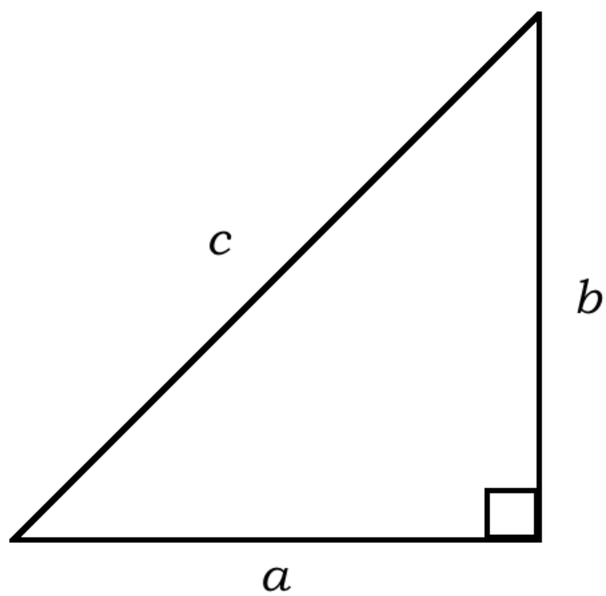
Try it yourself

Test the Pythagorean Theorem on the triangles below:



Applying the Pythagorean Theorem

Now that we understand the relationship between the the lengths of the sides for right triangles, we can solve for the length of any side if we are given the length of the two other sides:



If $a^2 + b^2 = c^2$, then:

1. $c = \sqrt{a^2 + b^2}$

2. $a = \sqrt{c^2 - b^2}$

3. $b = \sqrt{c^2 - a^2}$

Exercise: Prove that the above equations are true, given that $a^2 + b^2 = c^2$.

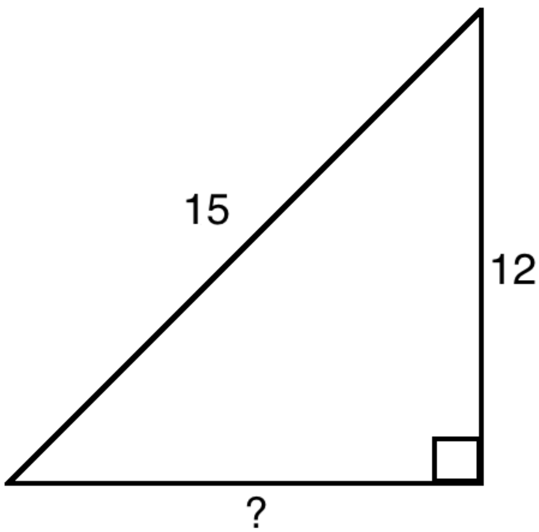
1.

2.

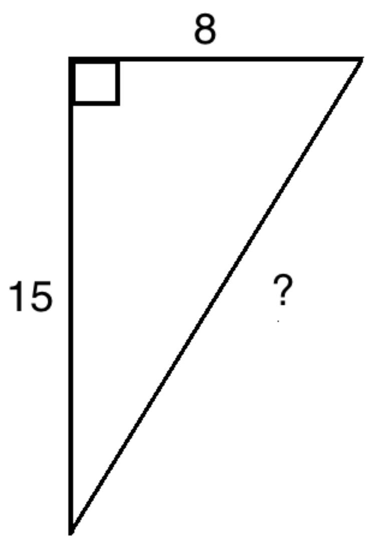
3.

Exercise

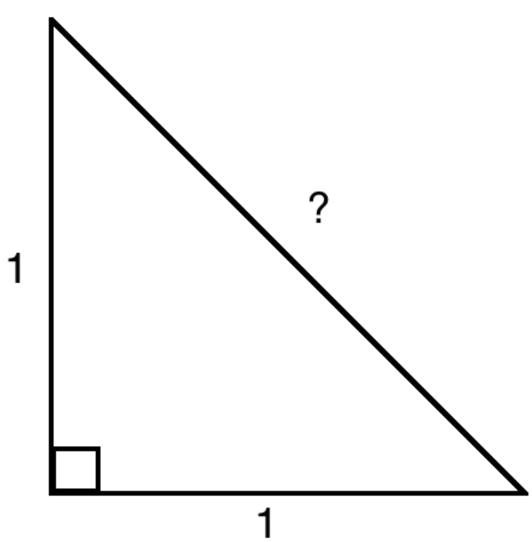
Solve for the length of the missing side. Assume the unit of measure in centimetres.



Let a be the length of the missing side. Solve for a.



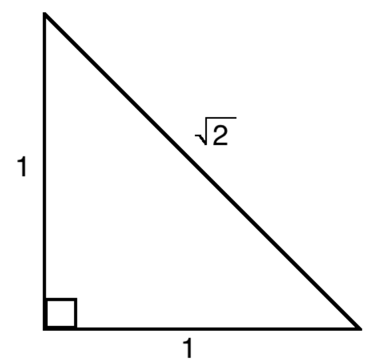
Let c be the length of the missing side. Solve for c.



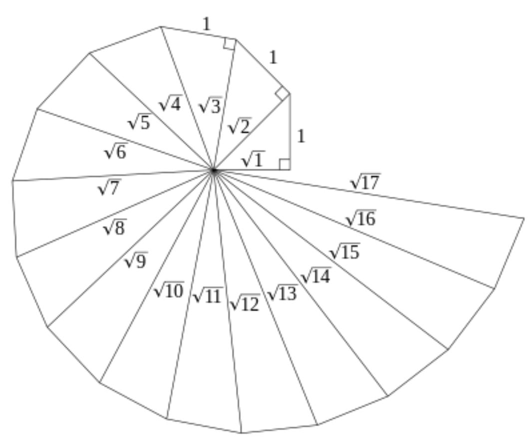
Let c be the length of the missing side. Solve for c.

Something interesting about $\sqrt{2}$

The last length we solved for in the exercise above was $\sqrt{2} = 1.4142135...$, which is an **irrational number-** an irrational number is just a number that we cannot express as a simple fraction, the same way we can express 0.5 as $\frac{1}{2}$. But, with the Pythagorean Theorem we can visually see the exact **distance** of $\sqrt{2}$.



In fact, there are many irrational numbers we can express as exact distances. **The Spiral of Theodorus** shows that we can visualize the exact distance of irrational numbers. The hypotenuse of each triangle becomes the leg of the next triangle in the spiral, with the shorter leg always having a distance of 1:

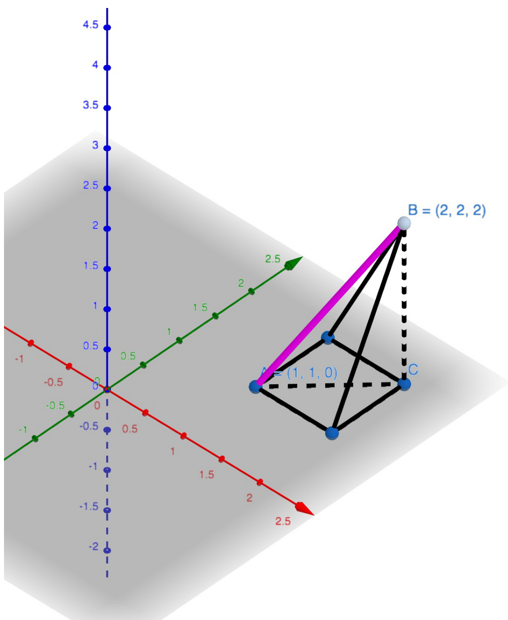


Picture from: http://jwilson.coe.uga.edu/EMAT6680Fa2012/Nelli/spiraloftheodorus/spiraloftheodorus.html

Distance functions

What is a distance function? A distance function takes two points as input and gives the distance between them as output. Pretty straightforward, right? Let's take a look at some.

You may or may not be familiar with the **absolute value** function. For any number a, the absolute value of a, written as $|a|$, is just the positive version of a. This is one way of thinking about size or distance. However, that only works in one dimension—on a line. If we want to find the distance something is away from something else in 2D, we use the Pythagorean Theorem and think about distance as the hypotenuse of the triangle defined by a point's co-ordinates. We don't need to stop ourselves at 2D, however. In fact, we can measure distance between any two points in 3D space using a **generalized version** of the Pythagorean Theorem that looks like $d^2 = a^2 + b^2 + c^2$. If we wanted to really be crazy, we could keep going to have as many spacial dimensions as we want. Whatever number of dimensions, we can always use the Pythagorean Theorem with a number of terms equal to the number of dimensions we're working in.



Example:

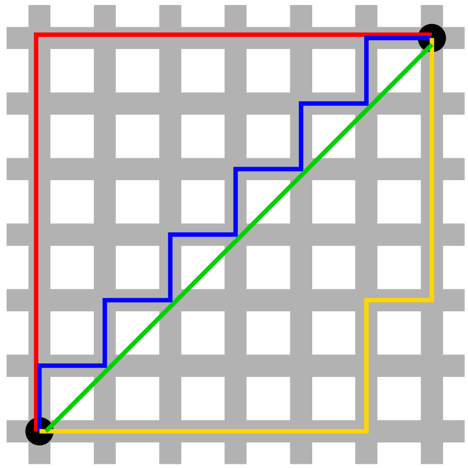
To find the distance between $A = (1, 1, 0)$ and $B = (2, 2, 2)$ on the left, we find the differences in their co-ordinates, and apply the Pythagorean theorem: $d = \sqrt{(2-1)^2 + (2-1)^2 + (2-0)^2} = \sqrt{1+1+4} = \sqrt{6} \approx 2.45$.

Try it yourself:

Find the distances between...

- $(-1, 5)$ and $(0, 3)$.
- $(2, -6)$ and $(3, 1)$.
- $(0, 1, 1)$ and $(3, 6, 0)$.
- $(4, 1, -3)$ and $(-5, 6, -1)$.
- $(1, 7, -3, 2)$ and $(3, -4, -6, -2)$.

However, the Pythagorean Theorem isn't the only distance function! In really advanced mathematics, there are all kinds of distance functions. However, we're only going to look at one more here: the **taxicab** or **Manhattan** distance. Imagine a grid like on a sheet of graph paper. Now imagine that we can only travel along the grid lines. There are no curves, no diagonals, only straight, perpendicular lines. Let the length of the squares be 1. Then the distance between two points is found by counting the number of unit lengths needed to get from one to the other. In the diagram below, the green line represents standard euclidean distance. What is the Manhattan distance between the two points pictured?



Problems

"*" indicates challenge question

1. Let a and b be the lengths of the legs of the right triangles below (in cm), and let c be the length of the hypotenuse (in cm). Solve for the missing length. Simplify your answers.

   (a) $a = 1$, $b = \sqrt{3}$, $c =?$

   (b) $a = 2$, $b = 2$, $c =?$

   (c) $a = 6$, $b =?$, $c = 10$

   (d) $a =?$, $b = 24$, $c = 25$

   (e) $a = 2$, $b = 4$, $c =?$

2. If a bird is on the ground, 5 metres away from the base of a tree, and the bird flies 13 metres to the top of the tree, how tall is the tree?

3. Denver is in a hot air balloon that has just taken off and is now floating 12 metres above its launching point. Regan is standing on the ground, 16 metres away from the launching point. How far apart are Denver and Regan?

4. Find the Manhattan distance between the following pairs of points

(a) $(1,4)$ and $(3,4)$

(b) $(1,2)$ and $(3,4)$

(c) $(-1,-4)$ and $(-3,4)$

(d) $(-2,3)$ and $(1,-2)$

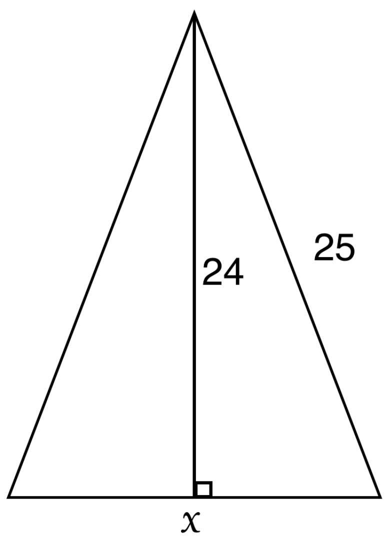
(e) $(3,-4)$ and $(-3,4)$

(f) $(6,-8)$ and $(-3,-1)$

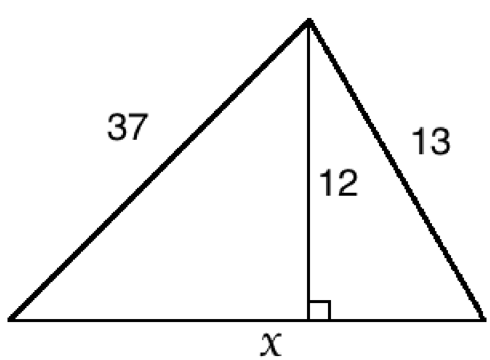
5. A circle is defined as all the points that are an equal distance from a centre. Using this definition, create a Manhattan circle with "radius" (this distance each point must be from the centre) four.

6. If the length of a rectangle is 24m and the length of the diagonal is 26m, what is the width of the rectangle.

7. Solve for the missing lengths.

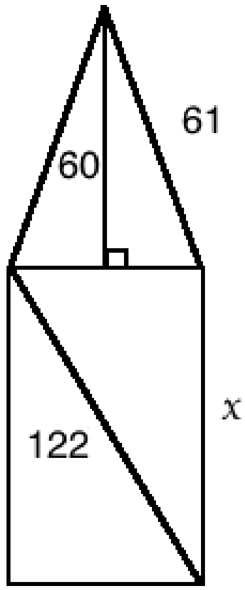


(a)

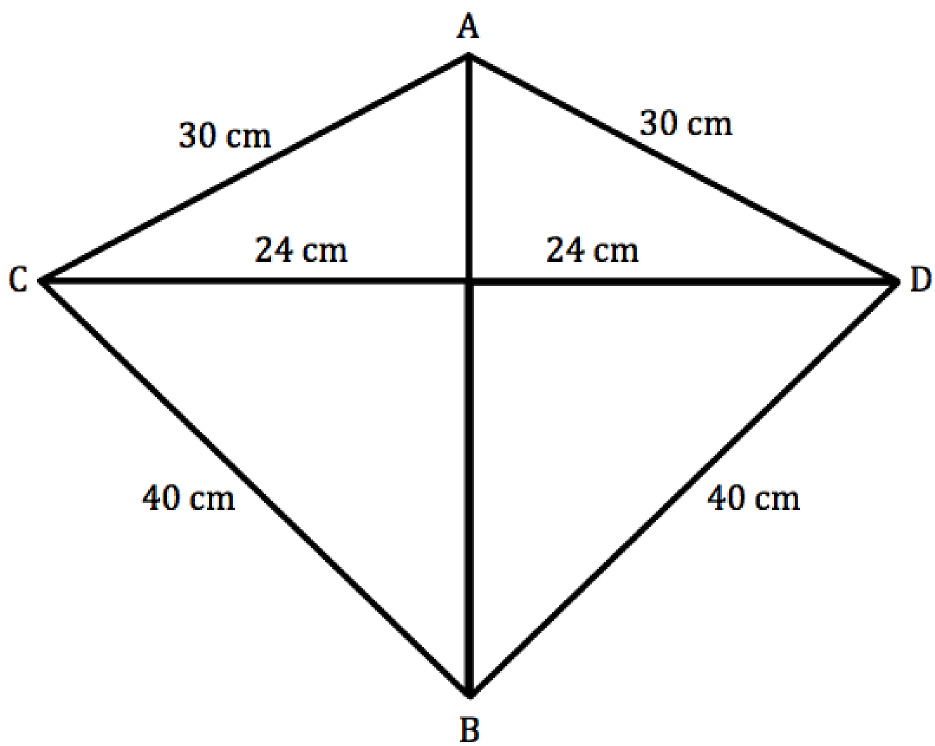


(b)

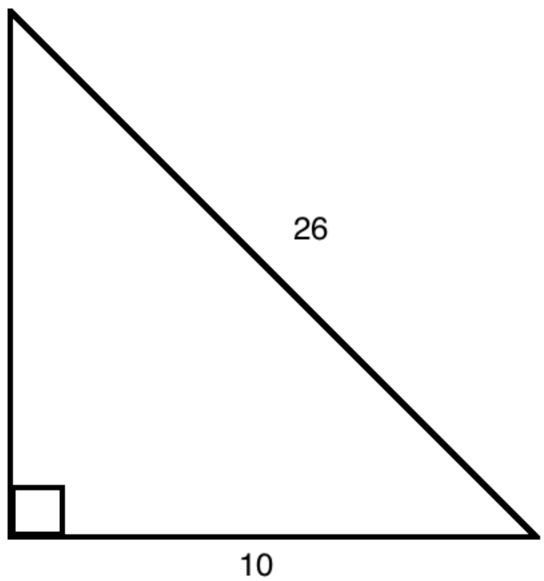
(c)



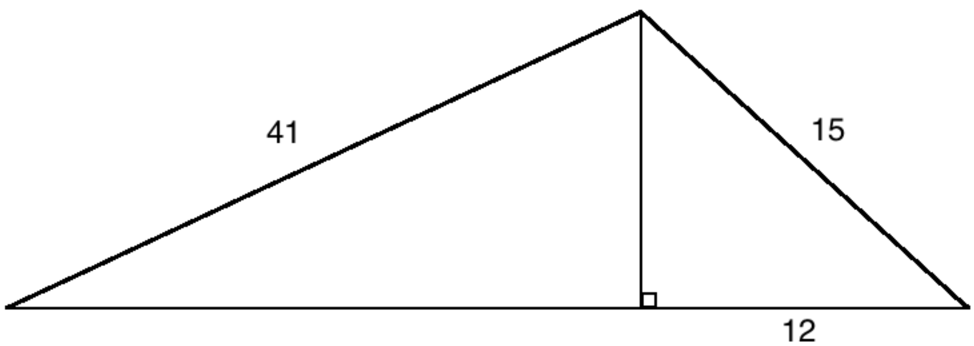
8. Solve for the length of AB



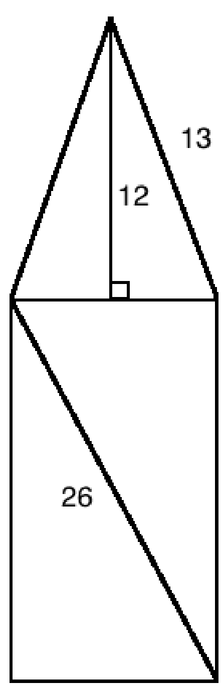
9. Solve for the area of the following shapes:



(a)



(b)

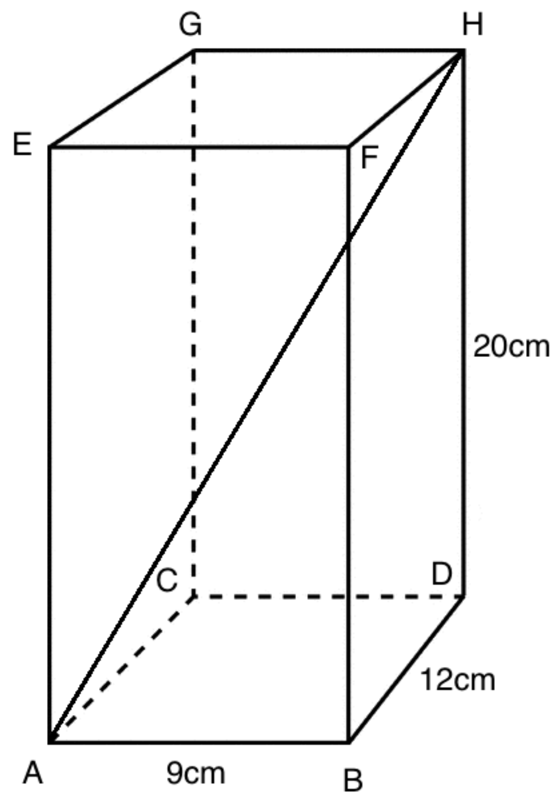


(c)

10. * If the area of of an isosceles triangle is $32cm^2$, and the base and height of the triangle are the same, what are the lengths of the other two sides.

11. * In the proof of the Pythagorean Theorem, use algebra to show that the area of the shaded region of the big square is equal to the area of the shaded region of the two smaller squares.

12. * Given the rectangular prism below, solve for the length of AH.



13. * A square-based pyramid has a height of $\sqrt{31}m$ and a base area of $100m^2$. What is the length of the slant of the pyramid (length from the corner of the base to the top of the pyramid)?

14. * Solve for the area of the trapezoid given the following information (round any calculations to one decimal place).