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# Grade 7/8 Math Circles

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***Ruler-and-Compass Constructions***

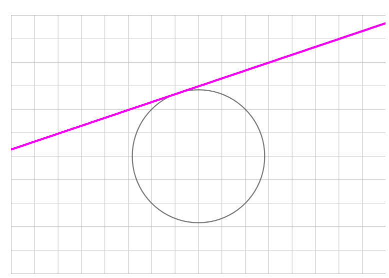
## Introduction

"Let no [person] ignorant of geometry enter here" – Plato, maybe [1]

Imagine that you have only two tools, an infinitely long straightedge which cannot be marked, and a compass which collapses upon removing it from the page. The straightedge ("ruler") may be used only to draw ____________ between two points, and the compass may be used only to draw ________ with radius given by two points. Such were the tools of the ancient Greeks in their practice of geometry. In fact, most Greek math was done using geometry and physical representations. They had no equations to represent objects, only algorithms and descriptions for drawing things which they knew to have certain properties. This means that when the Greeks wanted to figure a certain problem out, they had to figure out how to draw it. The most common way the Greeks did this was with ruler and compass as above.

## Simple constructions

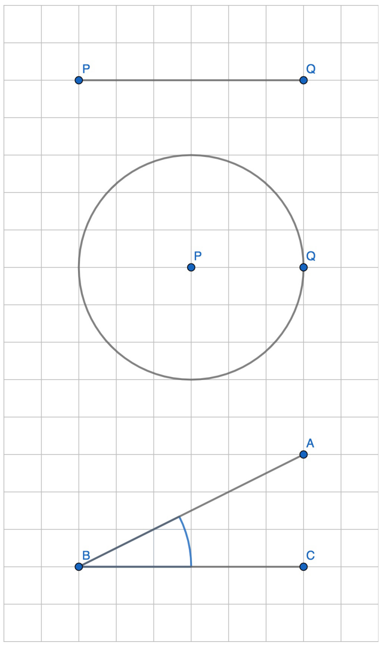
The Greeks, by the time they had finished their geometry, had learned how to construct regular polygons of three, four, and five sides, how to bisect (cut in half) any angle or line, how to construct a line tangent to a circle, and how to construct a polygon of double the number of sides of any given polygon. **Tangent** means that the line touches the circle exactly once. If we zoomed in far enough, it would look like the line and the circle were almost parallel. Today, we are going to go through the basics of ruler-and-compass construction.



We shall learn how to bisect lines, double the number of sides of a given polygon, and how to construct regular triangles, squares, and pentagons. There is also a special (though rather difficult) construction waiting for you among the problems.

[1] The earliest evidence we have that Plato actually wrote this comes from much after Plato, thereby diminishing its historical credibility. While it certainly would not be out of character for Plato, it is uncertain whether he did indeed write it.

# Reading instructions




The Greeks wrote down how to construct any of their geometry using the language of points, lines, and circles. Points are labeled with letter names like $P$ and $Q$, lines between two points are denoted $PQ$, and circles are described according to the position of their centre and their radius. Lengths of lines are denoted $|PQ|$. Sometimes intersections are described as "let (line) cut (line) in (point)," where the point is the intersection. Angles are labeled according to the three points involved, in order in-middle-out. For example, the angle on the left could be called either $\angle ABC$ or $\angle CBA$. Sometimes instructions will have the geometer skip simple steps, so we'll go through easier constructions here. Usually when constructing regular polygons, we construct them inside a circle.

One of the most useful tools you have at your disposal is the ability to create equidistant points (points that are equal distances away from a central object) and cut things in half. Let's start with the perpendicular bisector and angle bisector.

**Try it together – perpendicular bisector**

Create two points, $A$ and $B$. Draw $AB$. Now draw two circles with radii $|AB|$, one centred at $A$, and one centred at $B$. Label the intersection points of the circles $C$ and $D$. Then $CD$ is the perpendicular bisector of $AB$.

**Try it yourself – angle bisector**

For a challenge, skip these instructions and just try bisecting an arbitrary angle.

Draw any pair of non-parallel lines, intersecting at a point $A$. Now choose any point along either line other than $A$ and label it $B$. Draw a circle centred at $A$ with radius $|AB|$ and let it cut the other line (the one that doesn't contain $AB$) at $C$. Now draw two circles of radius $|BC|$, one centred at $B$, and one centred at $C$. Draw a line through the intersection points of the two circles. That line is the bisector of the angle $\angle CAB$.

Can you see how we used the perpendicular bisector to create an angle bisector? All we needed were two points, one on each line, that were the same distance apart so when we drew a line between them, that line's bisector went through the point we started with.

**Try it together – constructing a triangle**

Begin with a circle, whose centre is marked $O$. Make a line through $O$ so that it intersects the circle on the bottom at $A$ and on the top at $P_1$. Construct a circle with radius $|OA|$ centred at $A$. Label the points that this new circle intersects with the original circle as $P_2$ and $P_3$. Construct the lines $P_1P_2$, $P_1P_3$ and $P_2P_3$ to form an equilateral triangle.

**Try it yourself – constructing a square**

Begin with a circle whose centre is labeled $O$. Now draw a line through $O$ so that it intersects the circle on the right at $B$ and on the left at $A$. Bisect $AB$. Label the intersections of the bisector and the circle $C$ at the top and $D$ at the bottom. Construct $AC$, $CB$, $BD$ and $DA$ to form a square.

You may have already guessed how to create a regular polygon with twice as many sides as an existing polygon. If you haven't, we'll go through it now. Begin with any regular polygon inside a circle. Next, erect perpendicular bisectors to each side. Find the points at which the bisectors intersect with the circle and join those points to the existing points. The resulting polygon has double the number of sides of the original.

**Try it together – double sides**

Now that you know how to construct a square, construct a regular octagon.

Finally, we shall construct a regular pentagon. This is pretty tricky, so don't worry if you find it difficult.

**Try it yourself – constructing a pentagon**

Follow these steps:

1. Begin, as always, with a circle. Draw a line through the centre of the circle, $O$, and label the right point of intersection $A$ and the left point $B$.

2. Create a bisector perpendicular to $BA$ and label the top point of intersection with the circle to be $C$.

3. Also create a perpendicular bisector for the line $OA$ and label the midpoint of $OA$ as $D$.

4. Construct the line $CD$.

5. Now bisect the angle $\angle CDO$ (if this is confusing, ask the instructor). Label the intersection of the angle bisector you just drew with the line $OC$ to be $E$.

6. We now want a line perpendicular to $OC$ which passes through $E$.

   (a) To do this, draw a circle centred at $E$ and label the points of intersection between that circle and $OC$ to be $F$ on the bottom and $G$ on the top.

   (b) Draw two circles with radius $|FG|$, one centred at $F$ and one centred at $G$. Label the left point at which the two circles intersect to be $H$.

   (c) Now draw the line $EH$ and extend it to the edge of the circle, labeling the intersection to be $I$.

7. The line $CI$ is the length of the side of a regular pentagon. All we need to do now is find three more equally spaced points on the circumference of the circle.

8. To do this, draw a circle centred at $C$ with radius $CI$ and label the point it intersects the original circle to be $J$.

9. Do the same thing but centred at $J$, labeling the new intersection point $K$.

10. Finally, draw one last circle with the same radius as the others but centred at $K$, labeling the new intersection to be $L$.

11. Join the points you just made along the circumference of the circle to form a regular pentagon.

# Modern constructions

You may wonder why we haven't spoken much about geometry done by others. That is because almost no-one did this kind of geometry after the Greeks until Gauss in the 16th and 17th centuries! In 1801, Gauss developed a way to tell which regular polygons could be constructed at all.

A yet much greater number of polygons can be constructed if one is allowed to "mark one's ruler." Typically, this is actually fitting a tool in between constructed lines etc. If one allows this, then the regular heptagon (7-gon), as well as infinitely more regular polygons are possible. Play around with it and see what you can come up with. Now you know almost as much geometry as the Greeks did! If you want more problems outside of those provided below, there's a wonderful mobile app called "Euclidea" which is full of geometrical puzzles.

# Problems

1. Construct lines at the following angles:
   (a) 45° (*Hint: bisect a right angle*)
   (b) 60° (*Hint: what is the interior angle of an equilateral triangle?*)
   (c) 30° (*Hint: bisect a 60° angle*)
   (d) 135° (*Hint: what is 180°−45°?*)
   (e) 120° (*Hint: what is 180°−60°?*)
   (f) 105° (*Hint: start with 120° and go back by 15°*)

2. Fill in the gaps in your knowledge of Greek geometry by doing the following:
   (a) Construct a line which is parallel to another line. (*Hint: 90° + 90° = 180°*)
   (b) Construct a circle and a point outside that circle. Create a line tangent to the circle through that point. To do this, construct a line from the middle of the circle, $O$, to your arbitrary point, $P$. Bisect this line. From the midpoint of $OP$ labeled $M$, draw a circle with radius equal to half of $|OP|$. Label the top intersection point of the two circles to be $A$. Then the line $AP$ is tangent to the circle in the direction of $P$. (This also works with the bottom intersection point.)
   (c) Put down three arbitrary points, then construct a circle which passes through all of them. To do this, first turn your three points (call them $A$, $B$, and $C$) into a triangle. Next, construct a circle of radius greater than half the length of the triangle's longest side at each point. Connect the intersection points of the circles and label the point where these new lines cross to be $O$. Draw a circle

of radius $|OA|$ (or, equivalently, $|OB|$ or $|OC|$) centred at $O$. That circle should pass through each of your three initial points.

3. For a significant challenge (confusing in parts even to the instructor), attempt to follow the instructions below to create the first piece of ruler-and-compass innovation since the Greeks! *Note: this construction may be attempted with pencil and paper, but it often does not work out. GeoGebra is a free-to-use web tool that we recommend you use for this exercise. It can be found here: https://www.geogebra.org/geometry?lang=en.*

   (a) Begin with a circle whose centre is $O$ and draw a horizontal line through the middle. Label the rightmost point of intersection $A$ and the leftmost point $Z$.

   (b) Erect a perpendicular bisector to the line $ZA$, labeling the top point of intersection $B$.

   (c) Bisect $OB$ twice to get a point one quarter of the radius away from $O$ and label it $I$.

   (d) Construct the line $IA$.

   (e) Bisect the angle $\angle OIA$ twice, bisecting the half closer to the centre of the circle the second time. Take the intersection of the second bisector and $OA$ and label it $E$.

   (f) Construct a line at an angle of $45°$ to $IE$, facing $ZA$. Label the point that this new line intersects $ZA$ to be $F$.

   (g) Mark the midpoint of $FA$ as $M$ by bisecting it (don't worry about keeping the bisecting line).

   (h) Prepare to draw a circle centred at $M$ with radius $|FM|$, but don't. Instead, just mark where the circle *would* intersect with $OB$ as the point $K$.

   (i) Draw the left half of the circle with radius $|EK|$ centred at $E$. Label the point of intersection between the circle and $ZO$ to be $N_5$.

   (j) Erect a line through $N_5$ which is perpendicular to $ZO$. Label the point where this line intersects with the very first circle to be $P_1$.

   (k) Create a circle centred at $P_1$ With radius $|P_1A|$.

   (l) Proceed as with the pentagon to create 17 equally-spaced and labeled points. *Note: some points these circles create are close to existing points. Perhaps mark them in a different colour to avoid confusion.*

   (m) Join the points around the circumference that you just made and you have yourself a 17-gon or Heptadecagon.