Senior Math Circles
Euclidean Geometry
February, 2019

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Today’s Plan

Last week we:

- learned about the 5 postulates that serve as the foundation for Euclidean geometry,
- we learned how to do some constructions using only a compass and straightedge (and a pencil),
- and we learned how to prove that the constructions accomplish the task they were designed to accomplish.
- We also earned the “right” to use the midpoint tool and the perpendicular bisector tool in GeoGebra.

Today we will:

- Focus a little more on the proofs,
- earn the right to use a few more of the tools in GeoGebra,
- and investigate some challenging constructions.
Midpoint of a Line Segment, Revisited

Construction

Draw two circles with equal radius, one with centre A and one with centre B. The radius must be equal to the length of AB. Label the two points of intersection of the circles as P and Q.

Claim: PQ is the perpendicular bisector of AB.

Let’s prove this together on the board.
Angle Bisector Revisited

Construction

Draw circle with centre $B$, ensuring that the radius is less than both $AB$ and $AC$. The circle will intersect line segment $AB$ at a point $P$ and intersect line segment $BC$ at a point $Q$.

Draw the perpendicular bisector of line segment $PQ$ so that it crosses the point $D$.

Claim: $BD$ is the angle bisector of $\angle ABC$.
The Challenge

Given a line segment $AB$ and a point $C$ not on $AB$,

a) construct line through $C$ that is perpendicular to the line containing $AB$.

b) construct line through $C$ that is parallel to the line containing $AB$.

Tools (may not need all of these):

- The line segment tool
- The intersection tool
- The angle bisector tool
- The circle tool
- The midpoint tool
- The point tool
- The perpendicular bisector tool

Let’s take some time to discover the constructions now.
Perpendicular Line Construction

1. Draw circle centred at $A$ through $C$
2. Draw circle centred at $B$ through $C$
3. Label the other point of intersection of these two circles $D$

Claim: The line through $DC$ is perpendicular to the line through $AB$. 
Parallel Line Construction

1. Construct \( CD \) as above
2. Draw any circle centred at \( C \)
3. Label \( E \) and \( F \) as the points of intersection with this circle and the line \( CD \) extended.
4. Construct the perpendicular bisector of \( EF \)

Claim: This perpendicular bisector passes through \( C \) and is parallel to \( AB \).
Pythagorean Theorem

Before we try the next few challenges, let’s have some fun and prove the Pythagorean Theorem. There are actually many ways to do this, but we will use the method Euclid used. Consider this diagram:

Let’s continue on the board!
More Challenges!

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10. **Square by Adjacent Midpoints:** Give two points $A$ and $B$, construct a square $CDEF$ so that $A$ is the midpoint of $CD$ and $B$ is the midpoint of $CF$. 

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