Two Variable Linear Inequalities

Graph the following regions that satisfy the inequalities

1. \( x - 2y \geq 3 \)

Consider when \( x = 1 \) for \( x - 2y \geq 3 \).
1 - 2y ≥ 3
1 - 3 ≥ 2y
-2 ≥ 2y
\[
\frac{2y}{2} \leq \frac{-2}{2}
\]
y \leq -1
2. \( x - 2y \geq 3 \cap x - 2y \leq 6 \)

Consider \( x = -1 \) for \( x - 2y \leq 6 \).

\[
\begin{align*}
(-1) - 2y & \leq 6 \\
-2y & \leq 7 \\
\frac{-2y}{-2} & \geq \frac{7}{-2} \\
y & \geq \frac{-7}{2}
\end{align*}
\]

Note that we flip the inequality sign when dividing by -2.

3. \( 5x + 3y < 12 \cup x - 2y \leq 6 \)
Consider $x = 2$ for $5x + 3y < 12$.

\[
\begin{align*}
5(2) + 3y &< 12 \\
10 + 3y &< 12 \\
3y &< 12 - 10 \\
y &< \frac{2}{3}
\end{align*}
\]

4. $x - y < 5$
1. Consider when $x = 0$ for $x - y < 5$

- $0 - y < 5$
- $-y < 5$
- $y > -5$

5. $x + 2y > 6 \cap 2x - y \leq 4$
6. $3x - y \leq 12 \cap x + y < 5 \cap x - 2y > 4$

Consider $y = 0$ for $x + 2y > 6$,

$$x + 2(0) > 6$$
$$x > 6$$

Consider $x = 6$ for $2x - y \leq 4$

$$2(6) - y \leq 4$$
$$12 - 4 \leq y$$
$$8 \leq y$$
$$y \geq 8$$
Consider $x = 0$ for $3x - y \leq 12$

$3 \text{(a)} - y \leq 12$

$-12 \leq y$
$y > -12$

Consider $y = 0$ for $x + y < 5$

$x + 0 < 5$
$x < 5$

Consider $x = 0$ for $x - 2y > 4$

$0 - 2y > 4$
$-4 > 2y$
$y < -2$
More Absolute Values (Review)

Solve each of the following inequalities algebraically and graphically

1. \(|x - 7| + |x - 1| < 8\)
   Where are our "special" points?

   \[
   x - 7 = 0 \quad \quad x - 1 = 0
   \]
   \[
   x = 7 \quad \quad x = 1
   \]

   We know for \(1 < x < 7\) that \(|x - 7| + |x - 1| = 6\). To make sure \(|x - 7| + |x - 1| < 8\) we can’t be more than one away from 7 and one away from 1. Therefore \(0 < x < 8\) satisfy the inequality.

   Use your knowledge about absolute values to prove the following properties. 
   *Hint: cases are your friend.*

2. If \(a\) and \(b\) are any real numbers and \(b \neq 0\), then \(|\frac{a}{b}| = \frac{|a|}{|b|}\)

   In order to prove this we need to consider 4 cases.

   **Case 1** \([a \geq 0, \ b > 0]\)

   If \(a \geq 0, b > 0\) and \(\frac{a}{b} > 0\) we know \(|\frac{a}{b}| = \frac{a}{b}\)

   Since \(a \geq 0\) and \(b > 0\) we know \(|a| = a\) and \(|b| = b\)

   Thus \(|\frac{a}{b}| = \frac{a}{b} = \frac{|a|}{|b|}\)

   **Case 2** \([a \geq 0, \ b < 0]\)

   If \(a \geq 0, b < 0\) then \(\frac{a}{b} \leq 0\) and \(|\frac{a}{b}| = -\frac{a}{b}\)

   Since \(a \geq 0\) and \(b < 0\) we know \(|a| = a\) and \(|b| = -b\)

   Thus, \(|\frac{a}{b}| = -\frac{a}{b} = \frac{a}{-b} = \frac{|a|}{|b|}\)

   **Case 3** \([a < 0, \ b > 0]\)

   If \(a < 0\) and \(b > 0\), then \(\frac{a}{b} < 0\) and \(|\frac{a}{b}| = -\frac{a}{b}\)
Since $a < 0$ and $b > 0$, we know $|a| = -a$ and $|b| = b$

Thus, $\frac{a}{b} = -\frac{a}{b} = \frac{-a}{b}$

**Case 4** $[a < 0, b < 0]$

If $a < 0$ and $b < 0$, then $\frac{a}{b} > 0$ and $|\frac{a}{b}| = \frac{a}{b}$

Since $a < 0$ and $b < 0$, we know $|a| = -a$ and $|b| = -b$

Thus, $\frac{a}{b} = \frac{a}{b} = \frac{-a}{b}$

Therefore we know $|\frac{a}{b}| = \frac{|a|}{|b|}$ when $a$ and $b$ are real numbers and $b \neq 0$

3. If $a$ is a real number and $n$ is an integer, then $|a^n| = |a|^n$

To prove $|a^n| = |a|^n$ we will consider the cases when $a \geq 0$ and $a < 0$

Proof:

**Case 1** $[a \geq 0]$

If $a \geq 0$ then $a^n \geq 0$ and $|a| = a$

Thus $|a^n| = a^n = |a|^n$

**Case 2** $[a < 0]$

If $a < 0$, then $|a| = -a$

If $n$ is even, then $a^n > 0$ and $|a^n| = a^n$

With $n$ even $(-1)^n = 1$

Thus $|a^n| = a^n = (-1)^n a^n = (-a)^n = |a|^n$

If $n$ is odd, then $a^n < 0$ and $|a^n| = -a^n$

With $n$ odd $(-1)^n = -1$

Thus $|a^n| = -a^n = (-1)a^n = (-1)^n a^n = (-a)^n = |a|^n$

Therefore we know $|a^n| = |a|^n$ when $a$ is a real number and $n$ is an integer
Triangle Inequality

1. A triangle can be formed having side lengths 4, 5 and 8. It is impossible however, to construct a triangle with side lengths 4, 5 and 10. Using the side lengths 2, 3, 5, 7 and 11, how many different triangles with exactly two equal sides can be formed?

There are five cases to consider. Let x represent the third side length.
Case 1 [2,2,x]
Triangle inequality says

\[ x + 2 > 2 \]
\[ 2 + 2 > x \]
\[ x + 2 > 2 \implies x > 0 \]
\[ 2 + 2 > x \implies x < 4 \]
So 0 < x < 4.
Since we can only have two equal side lengths x = 3 is our only possibility.
Case 2 [3,3,x]
Similarly we can show 0 < x < 6.
Thus, the only possibilities for x are 2 and 5.
Case 3 [5,5,x]
Similarly we can show 0 < x < 10.
Thus the only possibilities for x are 2, 3 and 7.
Case 4 [7,7,x]
Similarly we can show 0 < x < 14.
Thus the only possibilities for x are 2, 3, 5 and 11.
Case 5 [11,11,x]
Similarly we can show 0 < x < 22.
Thus the only possibilities for x are 2, 3, 5 and 7.
Therefore 1 + 2 + 3 + 4 + 4 = 14 different triangles can be formed under the given conditions.