# Intermediate Math Circles
# October 23, 2019
# Counting Part 3 - More Counting

**Review**

A **permutation** of a set of distinct objects is an ordered arrangement of the objects. The number of permutations of $n$ objects is $n!$.

An ordered arrangement of $r$ elements from a set of distinct objects is called an $r$-**permutation**. The number of $r$-permutations from a set of $n$ distinct objects is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

The number of permutations of $n$ objects where there are $n_1$ indistinguishable type 1 objects, $n_2$ indistinguishable type 2 objects, ..., $n_k$ indistinguishable type k objects, where $n_1 + n_2 + \cdots + n_k = n$ is $\dfrac{n!}{n_1!\, n_2!\, ...\, n_k!}$.

An $r$-**combination** of a set of distinct elements is an unordered selection of $r$ elements from the set. The number of $r$-combinations from a set with $n$ distinct elements is $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$.

Example: How many ways can we select 4 blocks from a bag which contains red, green and blue blocks? (There are many blocks of each colour.)

$$\binom{4+2}{4} = \binom{6}{4} = \frac{6!}{4!2!} = \frac{6(5)}{2} = 15$$

The number of $r$-combinations from a set with $n$ elements when repetition of elements is allowed is $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$.

**Permutations with Repetition (unfixed number of each type)**

The number of $r$-permutations from a set with $n$ elements when repetition of elements is allowed is $n^r$.

Example: How many strings of length 5 can be formed using letters of the English alphabet?

$26^5 = 118\,813\,376$

**Distributing Objects into Boxes**
When distributing objects into boxes we need to consider whether the objects are distinguishable or indistinguishable and whether the boxes are distinguishable or indistinguishable (labelled or unlabelled).

**Distinguishable objects and distinguishable boxes**
**Example 1:** 15 different tasks need to be completed by 3 different people. How many different ways can we distribute the tasks among the 3 people so that each person gets 5 jobs?

$\binom{15}{5}\binom{10}{5}\binom{5}{5} = \dfrac{15!}{5!10!}\dfrac{10!}{5!5!}\dfrac{5!}{5!0!} = \dfrac{15!}{5!5!5!} = 756\,756$

The number of ways to distribute $n$ distinguishable objects into $k$ distinguishable boxes so that $n_i$ objects are placed into box $i$ (where $i = 1, 2, 3, \dots, k$) is $\dfrac{n!}{n_1!\, n_2!\, \dots\, n_k!}$.

**Example 2:** A student must select three books to read from a list of 50 different books. No two students are permitted to read the same book. How many ways can 15 students select their books?

$\dfrac{50!}{(3!)^{15}5!} \approx 5.39 \times 10^{50}$

**Indistinguishable objects and distinguishable boxes**
**Example 1:** How many ways can 20 identical doughnuts be distributed among 10 different people?

Using stars and bars method: $\binom{29}{9} = 10\,015\,005$

The number of ways to assign $n$ indistinguishable objects to $k$ distinguishable boxes is $\binom{n+k-1}{n}$. (This is an $n$-combination of a set with $k$ elements where repetition is allowed.)

**Example 2:** How many ways can some or all of 25 loonies be distributed among 7 people?

7 peopole + leftovers = 8 "boxes"

$\binom{25+8-1}{7} = \binom{32}{7} = 3\,365\,856$

**Problem Set 1**

1. How many 6-digit numbers have only odd digits?

2. A publisher has 3000 identical copies of a book. How many ways can they store the books at 3 different warehouses?

3. How many ways are there to deal hands of 7 cards to each of five players from a standard deck of 52 cards?

4. Ankur is distributing 16 different photos to his uncle's family. He wants to give each adult twice as many photos as those given to each child. If there are 4 children and 2 adults in his uncle's family, in how many ways can he distribute the photos?

5. How many solutions are there to the equation $a + b + c + d + e = 22$

   (a) where each value is a non-negative integer?

   (b) where each value is an integer greater than 1?

**(In)distinguishable objects and indistinguishable boxes**
There is no simple closed formula for the number of ways to distribute $n$ objects (whether distinguishable or indistinguishable) into $k$ indistinguishable boxes.

**Example 1:** How many ways can we place 5 copies of the same book into 4 identical boxes where a box can contain up to 5 books?

5, 41, 32, 311, 221, 2111 so there are 6 different ways.

**Example 2:** How many ways can we place 5 different books into 4 identical boxes where a box can contain up to 5 books?
For each of the ways above we need to determine which book goes in each box.

- 5 - 1 way
- 41 - $\binom{5}{4} = 5$ ways
- 32 - $\binom{5}{3} = 10$ ways
- 311 - $\dfrac{\binom{5}{3}\binom{2}{1}}{2!} = 10$ ways
- 221 - $\dfrac{\binom{5}{2}\binom{3}{2}}{2!} = 15$ ways
- 2111 - $\dfrac{\binom{5}{2}\binom{3}{1}\binom{2}{1}}{3!} = 10$ ways

In total: $1 + 5 + 10 + 10 + 15 + 10 = 51$ ways.
Distributing $n$ indistinguishable objects into $k$ indistinguishable boxes is the same as writing $n$ as the sum of at most $k$ positive integers in non-increasing order. If $a_1 + a_2 + \cdots + a_j = n$ where $a_1, a_2, \ldots, a_j$ are positive integers with $a_1 \geq a_2 \geq \cdots \geq a_j$, we say that $a_1, a_2, \ldots, a_j$ is a **partition** of positive integer $n$ into $j$ positive integers.

To count the number of ways of distributing distinguishable objects into indistinguishable boxes you need to sum values known as **Stirling numbers of the second kind**.

**Pigeonhole Principle** Let $N$ and $k$ be two positive integers where $N > k$. If $N$ objects are placed into $k$ boxes, then there is at least one box which contains at least two objects.

**Example 1:** What is the least number of people needed to guarantee that two people have the same birthday?

367

**Example 2:** You have 15 different pairs of socks in a drawer and you randomly draw one sock at a time. How many socks must you draw before you are guaranteed to have a matching pair?

16

**Problem Set 2**

1. How many ways can you place 17 identical laptops into 5 identical bags such that each bag has at least 2 laptops?
2. How many ways can we place 7 different pictures into 5 identical folders?
3. How many random English words can you write down before you are guaranteed to have two words which have the same first letter and the same last letter?
4. You are placing 27 apples into 4 baskets. Find the largest integer $k$ such that you are guaranteed to have a basket with at least $k$ apples in it.
5. How many ways can you place a dozen different books on four distinguishable shelves if the order of the books on the shelves matter?