Problem Set 1 - Solutions

Intermediate Math Circles Fall 2019
Fun With Inequalities

October 30, 2019

Linear Inequalities- Single Variable

Solve each of the following.

1. $x + 5 < \frac{7}{2}$

$$\begin{aligned} x &< \frac{7}{2} - 5 \\ x &< \frac{7}{2} - \frac{10}{2} \\ x &< \frac{-3}{2} \end{aligned}$$

Therefore $x < \frac{-3}{2}$ satisfies the inequality.

2. $3 - \frac{x}{2} \geq -8$

$$\begin{aligned} -\frac{x}{2} &\geq -8 - 3 \\ -\frac{x}{2} &\geq -11 \\ -2(\frac{-x}{2}) &\leq -2(-11) \end{aligned}$$

We multiply by -2, so we flip the inequality.

$$x \leq 22$$

Therefore $x \leq 22$ satisfies the inequality.

3. $-1-3x \leq 4x+10$

$$\begin{aligned}-3x-4x &\leq 1-+1\\ -7x &\leq 11\\ \frac{-7x}{-7} &\geq \frac{11}{-7}\end{aligned}$$

Since we are dividing by -7, we flip the inequality.

$$x \geq \frac{-11}{7}$$

Therefore $x \geq \frac{-11}{7}$ satisfies the inequality.

4. $2x+5>4x-7$

$$\begin{aligned}5+7 &> 4x-2x\\ 12 &> 2x\\ \frac{12}{2} &> \frac{2x}{2}\\ 6 &> x\\ x &< 6\end{aligned}$$

Therefore $x<6$ satisfies the inequality.

5. $-\frac{2}{3}x+\frac{3}{7} \leq 5-\frac{x}{2}$

We clear the fractions by multiplying the LHS and the RHS by 42. We use 42 because it is the lowest common multiple.

$$\begin{aligned}42(\frac{-2x}{3}+\frac{3}{7}) &\leq 42(5-\frac{x}{2})\\ -28x+18 &\leq 210-21x\\ 18-210 &\leq -21x+28x\\ -192 &\leq 7x\\ \frac{7x}{7} &\geq \frac{-192}{7}\\ x &\geq \frac{-192}{7}\end{aligned}$$

Therefore $x \geq \frac{-192}{7}$ satisfies the inequality.

Properties

1. Which of the eight properties of $\leq$ also hold for $<$?

   The following properties hold for $<$ as well:
   (3) If $x < y$ and $y < z$, then $x < z$
   (4) One of the following three holds:
   $x < y$, $y < x$, or $x = y$
   (5) If $x < y$, then $x + r < y + r$
   (6) If $x < y$ and $r > 0$, then $rx < ry$
   (7) If $x < y$ and $4 < 0$, then $ry < rx$

2. Use whichever of the properties (1) to (8) that you need to prove the following

   (a) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.

   Proof
   We know $a + c \leq b + c$ by property 5.
   We also know $b + c \leq b + d$ by property 5.
   Then by transitivity (property 3) we know $a + c \leq b + c \leq b + d$.
   Thus $a + c \leq b + d$.

   (b) If $0 \leq a \leq b$ and $0 \leq c \leq d$, then $0 \leq ac \leq bd$.

   Proof
   We know $a \leq b$ and $c \geq 0$.
   Then by property 6 we know $ca \leq cb$.
   Similarly, we know $bc \leq bd$ by property 6.
   Then by transitivity (property 3) we know $ca \leq cb = bc \leq bd$.
   Thus $ca \leq bd$.

3. (a) If $a \leq b$ and $c \leq d$, is it true that $ac \leq bd$?

   No! Consider the following counterexample:
   Let $a = -7$, $b = -1$, $c = -5$ and $d = 3$.
   Clearly $a \leq b$ and $c \leq$ d.
   But $ac = (-7)(-5) = 35$ and $bd = (-1)(3) = -3$.
   That's a problem because $bd \leq ac$ as $-3 \leq 35$.

   (b) If $a \leq b$, is it true that $\frac{1}{b} \leq \frac{1}{a}$?

   No! Consider the following counterexample:
   Let $a = -1$ and $b = 1$.

Thus $\frac{1}{a} = -1$ and $\frac{1}{b} = 1$.
But $-1 \leq 1$.
<u>Note:</u> the statement also hasn't dealt with the possibility of $a = 0$ or $b = 0$.

4. Show that if $a < b$, then $a < \dfrac{1}{2}(a + b) < b$.

   If $a < b$, we know by property 5 that

$$\begin{aligned} a + a &< a + b \\ \frac{2a}{2} &< \frac{a+b}{2} \\ a &< \frac{a+b}{2} \end{aligned}$$

   We also know by property 5

$$\begin{aligned} a + b &< b + b \\ \frac{a+b}{2} &< \frac{2b}{2} \\ \frac{a+b}{2} &< b \end{aligned}$$

   Then by transitivity (property 3) we know
   $a < \frac{a+b}{2} < b$.

5. Show that the sum of a positive number and its reciprocal is at least 2. In other words show that

$$a + \frac{1}{a} \geq 2$$

   We know by property 8 that

$$\begin{aligned} (a-1)^2 &\geq 0 \\ a^2 - 2a + 1 &\geq 0 \\ a^2 + &\geq 2a \end{aligned}$$

   Since $a > 0$ we can divide both the LHS and the RHS by a. That is

$$\begin{aligned} \frac{a^2+1}{a} &\geq \frac{2a}{a} \\ \frac{a^2}{a} + \frac{1}{a} &\geq 2 \\ a + \frac{1}{a} &\geq 2 \end{aligned}$$

6. If $a \geq b$ and $c \geq d$, is it true that $a - c \geq b - d$?
No! Consider the following counterexample:
Let $a = 5$, $b = 4$, $c = 10$ and $d = 5$.
Clearly $a \geq b$ and $c \geq d$
But $a - c = 5 - 10 = -5$ and $b - d = 4 - 5 = -1$.
That's a problem because $b - d \geq a - c$ as $-1 \geq -5$.

Word Problems

1. Write this mathematical sentence using algebra:
6 is subtracted from a number x. Then the result is multiplied by 4. The final answer is less than or equal to 15.
A. $6(x - 4) \leq 15$
B. $4(x - 6) < 15$
C. $4(x - 6) \leq 15$
D. $4x - 6 \leq 15$

The answer is C. In other words, this mathematical sentence using algebra is $4(x - 6) \leq 15$.

2. Sandy and Mandy play in the same soccer team. Last Saturday Sandy scored 4 more goals than Mandy, but together they scored less than 12 goals. What are the possible number of goals Sandy scored?

Assign Letters:
the number of goals Sandy scored: S
the number of goals Mandy scored: M
We know that Sandy scored 4 more goals than Mandy did, so: $S = M + 4$
And we know that together they scored less than 12 goals: $S + M < 12$
We are being asked for how many goals Sandy might have scored: S

SOLVE: Start with: $M + S < 12, \quad S = M + 4$,
so: $M + (M + 4) < 12$
Simplify: $2M + 4 < 12$
Subtract 4 from both sides: $2M < 12 - 4$
Simplify: $2M < 8$
Divide both sides by 2: $M < 4$
Mandy scored less than 4 goals, which means that Mandy could have scored 0, 1,2 or 3 goals.
Sandy scored 4 more goals than Mandy did, so Sandy could have scored 4, 5, 6, or 7 goals.

Check:
When $M = 0$, then $S = 4$ and $S + M = 4$, and $4 < 12$ is correct
When $M = 1$, then $S = 5$ and $S + M = 6$, and $6 < 12$ is correct
When $M = 2$, then $S = 6$ and $S + M = 8$, and $8 < 12$ is correct
When $M = 3$, then $S = 7$ and $S + M = 10$, and $10 < 12$ is correct
(But when $M = 4$, then $S = 8$ and $S + M = 12$, and $12 < 12$ is incorrect)

3. Of 12 puppies, there are more girls than boys. How many girl puppies could there be? We assume that there is at least one boy puppy.

   Assign Letters:
   the number of girls: g
   the number of boys: b
   We know that there are 12 pups, so: $g + b = 12$, which can be rearranged to $b = 8g$
   We also know there are more girls than boys, so: $g > b$
   We are being asked for the number of girl pups: g
   Solve:
   Start with: $g > b$
   $b = 12 - g$, so: $g > 12 - g$
   Add g to both sides: $g + g > 12$
   Simplify: $2g > 12$
   Divide both sides by 2: $g > 6$
   So there could be 7, 8, 9, 10, or 11 girl pups.

   Check:
   When g = 7, then b = 5 and $g > b$ is correct
   When g = 8, then b = 4 and $g > b$ is correct
   When g = 9, then b = 3 and $g > b$ is correct
   When g = 10, then b = 2 and $g > b$ is correct
   When g = 11, then b = 1 and $g > b$ is correct

4. Alex went to the carnival with 27.50 dollars. He bought a hot dog and a drink for 6.50 dollars, and he wanted to spend the rest of his money on ride tickets which cost 1.50 dollar each. What is the maximum number of ride tickets that he can buy?

   Let $x =$ the number of ride tickets he can buy.
   Then the cost of food + the cost of rides ≤ 27.50
   $6.50 + 1.50x \leq 27.50$
   $1.50x \leq 27.50 - 6.50$
   $1.50x \leq 21$ [dividing both sides by 1.50]
   $x \leq 14$
   Alex can buy a maximum of 14 ride tickets.

Acknowledgement

The word problem section is adapted from `https://www.mathsisfun.com/algebra/inequality-questions-solving.html`