UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

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Probability

Probability is the study of how likely an event is to occur. Probability is used commonly in many applications so understanding how we can use it to predict outcomes is very important.
Probability is used in:

- Weather Forecasts
- Sports: batting averages, free throw percentage, field goal percentage
- Lottery
- Medical decisions: operation success rate

There are different types of probability:
Theoretical Probability is predicting the outcome of a situation with some logical reasoning, and/or a known formula to calculate the probability of an event happening.
Example: When flipping a coin you expect to get heads $\frac{1}{2}$ *the time.*

Experimental Probability is an estimate for the likelihood of an event based on data collected during experiments, direct observation, experience, or practice.
Example: After flipping a coin 50 times, you get heads 27 times. You would say you by the experimental probability you get heads $\frac{27}{50}$ *times when flipping a coin.*

As most of us are not experimentalists we will work with Theoretical Probability.

Probabilities are often represented by fractions or decimals with values between 0 and 1:

$$\frac{3}{6} = \frac{1}{2} = 0.5 \qquad \frac{1}{3} \approx 0.33 \qquad \frac{13}{52} = \frac{1}{4} = 0.25$$

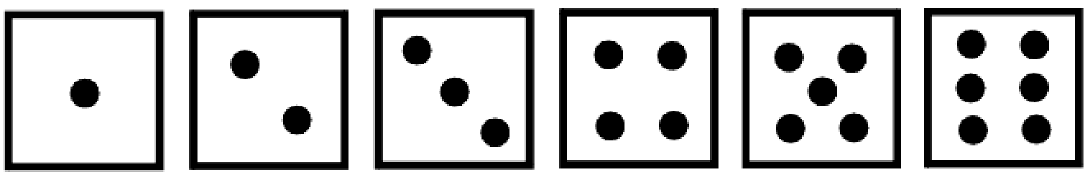
We say that the probability of any event **A** occuring is:

$$\text{Probability of A} = \text{P(A)} = \frac{\textit{number of ways } \mathbf{A} \textit{ can occur}}{\textit{total possible outcomes}}$$

Let's explore this equation more:

The **Sample Space** of an activity refers to the set of all possible outcomes for that activity.

For example all possible outcomes of rolling a six-sided regular die are: **{1,2,3,4,5,6}**.



When I roll a die I can only have 6 possible outcomes. 6 would be the denominatior in the formula for the probability above. The sample space would be much larger if you were to draw a card from a deck. Now the sample space would be the set of all 52 cards in the deck.

Exercises:

1. You roll a six-sided die a single time.

   (a) What is the probability of rolling a 2?

   (b) What is the probability of rolling a number that is even?

   (c) What is the probability of rolling a prime number?

   Hint: *A* ***prime*** *is an integer greater than 1 such that its only positive divisors are 1 and itself (e.g. 2, 3, 5, 7, 11, ...).*

   (d) What is the probability of rolling a 7?

2. You draw a single card from a deck of 52 cards.

    (a) What is the probability to draw the 2 of spades?

    (b) What is the probability to draw a diamond?

    (c) What is the probability to draw an Ace?

3. There are 6 beads in a bag, 3 are red, 2 are yellow and 1 is blue. What is the probability of picking a yellow? What is the sample space of the bag?

Relationships

There are three main relationships between events that we will be working with. It is important to recognize the difference because the relationship of two events affects how we calculate probabilities

- **Independent Events:** Two events such that the occurrence of one does not affect the probability of the other.

  Ex: *Drawing a Jack from a standard deck of cards AND getting a head from a coin-toss*

- **Dependent Events:** Two events such that the occurrence of one affects the probability of the other.

  Ex: *Stealing a car increases your chances of going to jail.*

- **Mutually-Exclusive Events:** Two events that cannot occur at the same time.

  Ex: *When flipping a coin it is impossible to get a head AND a tail on a single flip*

Exercises:

1. State whether the following pairs of events are Mutually-Exclusive, Independent or Dependent:

   (a) Drawing a 7 from a deck of cards AND drawing a Jack after (without replacement).

   (b) Drawing a 7 AND an 8 from a deck in one draw.

   (c) Getting a head on a coin-toss AND getting a tail on a different coin-toss.

   (d) Flipping a coin AND rolling a 6 on a die.

Multiple Events: Counting

Product Rule - Fundamental Counting Principle (FCP)

If you have to make Choice A ***AND*** *Choice B and there are m options for Choice A, n options for Choice B, then the total number of ways you can make Choice A* ***AND*** *Choice B is $m \times n$.*

Example: The ice cream truck sells ice creams in sizes regular and large in one of the 5 flavours: chocolate, vanilla, strawberry, peanut butter and mango. How many different ice creams can they make in total?

Solution: In other words, how many ice creams can they make if they have 2 sizes and 5 flavours?

$$2 \text{ sizes} \times 5 \text{ flavours} = \textbf{10 ice creams in total}$$

Sum Rule

If you have to make Choice A ***OR*** *Choice B and there are m options for Choice A, n options for Choice B, then the total number of ways you can make Choice A* ***OR*** *Choice B is $m + n$.*

Example: The Grade 5 and 6 students of a school are asked to choose whether they prefer dogs or cats. Of the 48 Grade 5 students, 36 prefer dogs and 12 prefer cats. Of the 52 Grade 6 students, 20 prefer dogs and 32 prefer cats. How many in total prefer dogs?

Solution: All students are either in Grade 5 **OR** Grade 6. The total number of students who prefer dogs are the number of Grade 5 students who prefer dogs plus the number of Grade 6 students who prefer dogs as follows: 36 G5 students prefer dogs + 20 G6 students prefer dogs = **56 total students prefer dogs**

Intersection and Union

Intersection

For two events A and B, the ***intersection*** *of A and B, denoted* $(A \cap B)$, *is the event that both A* ***AND*** *B happen.*

Exercise: You roll a 6-sided die. Let event A be that the number rolled is greater than 2. Let event B be that the number rolled is odd. What is the **intersection**, $(A \cap B)$ of the two events?

Union

For two events A and B, the ***union*** *of events A and B, denoted* $(A \cup B)$, *is the event that A* ***OR*** *B occurs.*

Exercise: You draw a card from a standard deck. Let event A be that the card is a heart. Let event B be that the card is a Queen. What is the **union**, $(A \cup B)$ of the two events?

Practice: If events A and B are mututally exclusive what is the intersection $(A \cap B)$ of the two events?

Compund Probabilities: Multiple Events

Sum Rule

Special Sum Rule

If you have to make Choice A ***OR*** *Choice B such that A and B are* ***mutually exclusive*** *and there are m options for Choice A, n options for Choice B, then the total number of ways you can make choices A* ***OR*** *B is $m + n$.* Additionally,

$$P(A \cup B) = P(A) + P(B)$$

Example: Given a bag of 3 blue balls, 5 red balls, 6 yellow balls and 6 green balls, what is the probability of picking a red ball or a green ball?

General Sum Rule

If you have to make Choice A ***OR*** *Choice B and there are m options for Choice A, n options for Choice B, then the total number of ways you can make choices A* ***OR*** *B is $m + n$ - $|A \cap B|$ where $|A \cap B|$ is the number of items in both A and B. Additionally,*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example: The numbers 1 to 20 are written on a paper and placed in a bag. What is the probability of picking a number divisible by 2 or a number divisible by 3?

Compund Probabilities: Multiple Events

Product Rule

Product Rule

For two events A and B, the probability of event A ***AND*** *event B* is as follows:

$$P(A \cap B) = P(A) \times P(B)$$

Example - Independent: A = rolling a 4 on a 6-sided die. B = flipping heads when flipping a coin. What is the probability of A and B?

Solution: We know that $P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{2}$. Then:

$$P(A \cap B) = P(A) \times P(B) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}.$$

Example - Dependent: A goblet contains 3 red balls, 2 green balls, and 6 blue balls. We choose a ball, then another ball without putting the first one back, what is the probability that the first ball will be red and the second will be blue?

Solution: Let A = first ball will be red and B = second ball will be blue.
$P(A) = \frac{3}{11}$ as there are 3 red balls available to choose and there are 11 balls in total to choose from. However, once we pick the first ball without putting it back, there will only be 10 balls left in the goblet. Since we want A **and** B, we can assume the first ball was red and so there are still 6 blue balls left to choose from of the 10 balls left. So given A, $P(B) = \frac{6}{10}$. Then:

$$P(A \cap B) = P(A) \times P(B) = \frac{3}{11} \times \frac{6}{10} = \frac{9}{55}.$$

Exercises

1. **Dependent**: A table of 5 students has 3 seniors and 2 juniors. The teacher is going to pick 2 students one after another at random from this group to present homework solutions. Find the probability that both students selected are juniors.

2. You draw a card from a deck of 52 cards.

   (a) What is the probability of drawing a 5 or a spade?

   (b) What about the probability of drawing a heart or a spade?

   (c) Find the probability of drawing two cards one after another and having them both be hearts (without replacement).

3. You roll two 6-sided dice.

   (a) Find the number of possible outcomes in the sample space.

   (b) What is the probability that both die show 1?

   (c) What is the probability that the sum of the two dice is an odd number that is greater than 5?

   (d) What is the most likely sum to occur from the roll?

Die 1

Die 2	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Complement

By now we know that the chance of rolling a 2 on a 6-sided die is $\frac{1}{6}$. What about the chance of rolling a number 1-6?

$$\frac{number\ of\ ways\ \mathbf{A}\ can\ occur}{total\ possible\ outcomes} = \frac{6}{6} = 1$$

Here we say that our probability is 1 or 100%. Our event of rolling a number 1-6 is equal to the entire sample space {1,2,3,4,5,6} or the total possible outcomes so it will always occur. This result tells us something important.

The sum of the probabilities of each event occuring in a discrete sample space is 1.

For the example of a six sided die above the sum of the probabilites to roll each number on the die would look like:

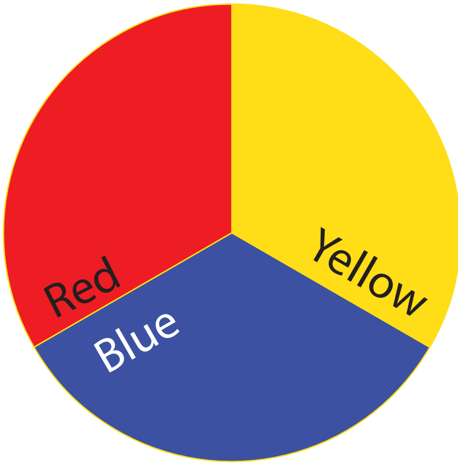
$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

This useful property of each discrete sample space allows to find what we call complement probabilities.

Complement

The ***complement*** *of an event* ***A*** *is the set of outcomes in the sample space not included in event* ***A****. The complement of* ***A*** *is written* $\bar{A}$*. We can say that* $P(A) + P(\bar{A}) = 1$ *since event* ***A*** *and all other events in the sample space that are not* ***A*** *make up the total possible outcomes. Thus* $P(\bar{A}) = 1 - P(A)$*.*

Examples:



1. What is the probability of spinning the above spinner and it landing on yellow? What is the complement of this probability?

Sometimes it can be useful when finding the probabilities of certain events to find the probability of their complements instead. We can also rearrange the complement equation as $P(A) = 1 - P(\bar{A})$.

2. A card is chosen at random from a deck of cards. What is the probability that the card is not a Jack?

Conditional Probability

Conditional Probability tells us the probability that one event will occur given that another one has already happened.

The likelihood of an event **B** occuring, given that event **A** has already happened. This probability is written as:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

Where:

- $P(B \mid A)$ represents the probability of event B given event A has already happened
- $P(B \cap A)$ is the probability of event A **AND** event B occuring
- $P(A)$ is the probability of event A occuring.

Examples:

Find the Probability of event B given that event A occurs and also has a 0.45 (45%) probability to occur. The probability of event A and event B occuring is 0.2 (20%.)

In a group of 100 pet owners, 40 own cats, 30 own dogs, and 20 own a dog and a cat. If a pet owner chosen at random owns a cat, what is the probability they also own a dog? What is the probability they do not own a dog?

Problem Set:

1. Find the following probabilities

   (a) What is the probability of getting a head or tail when flipping a coin?

   (b) What is the probability of rolling a 7 on a 6-sided die?

   (c) What is the probability of drawing a Jack from a deck of cards?

   (d) The probability that there is a Math Circles lesson today is 3/7. What is the probability that there is not a Math Circles lesson?

   (e) You time travel to a random day in the next leap year. What is the probability that the day is Christmas?

2. Identify the following pairs of events as independent, dependent, or mutually exclusive.

   (a) Driving fast and getting a speeding ticket.

   (b) Picking a diamond from a deck of cards and then drawing another diamond without replacing the first card.

   (c) Picking a diamond and then picking a spade from a deck of cards (without replacement).

   (d) Flipping a coin and getting heads and then drawing the 3 of spades form a deck of cards.

   (e) A student being in math circles and having interest in math.

   (f) Flipping a coin and getting heads and tails.

   (g) Picking two red balls consecutively (back to back) when picking from a bag containing 10 red, 12 black and 2 white balls (with replacement).

3. Two 6 sided die are rolled. What is the probability that:

   (a) their sum is a prime number?

   (b) their sum is not a prime number?

   (c) they both show the same number?

   Hint: *Use the chart showing the possible outcomes when rolling two die for part (a) and (b).*

4. You have a bag with 13 purple and 24 yellow marbles. 5 of the purple marbles are large and the rest are small. 9 of the yellow marbles are large and the rest are small. What is the probability of each of the following:

   (a) picking a yellow marble?

   (b) picking a small marble?

   (c) picking a purple and small marble?

   (d) picking a yellow or large marble?

5. Find the following Probabilities:

   (a) What is the probability that I roll a 3 on a die twice in a row?

   (b) What is the probability that I draw a Queen of Hearts twice in a row (I replace the first Queen and draw again)?

6. What is the probability of **winning** (ties do not count as winning) a game of Rock, Paper, Scissors? (assume each of rock, paper, and scissors is just as likely to be chosen.)

   (a) What is the probability to not win a game of rock paper scissors?

   (b) Your friend argues that you are more likely to get heads twice in a row on two consecutive coin flips than not win three games of rock paper scissors in a row. Is your friend correct?

7. A weighted coin *(it is no longer fair)* is altered so that the probability of it landing on a head for each flip is $\frac{5}{7}$. The trick coin is flipped 3 times. What is the probability that head appears on the first flip and tail appears on the last flip?

8. The Ministry of Magic is holding a lottery and has sold 2000 tickets. If Harry Potter has a $\frac{1}{16}$ chance of winning, then how many tickets did he purchase?

9. Answer the following questions using conditional probability:

   (a) The probability that it is Monday and that a student is absent is 0.12. Since there are 5 school days in a week, the probability that it is Monday is 0.2. What is the probability that a student is absent given that today is Monday?

   (b) Given that events **A** and **B** are mutually exclusive, without performing any calculations, find $P(A \mid B)$.

(c) In your math class, 30% of the students passed both tests on the probability unit and 45% of them passed the first test. What percent of students that passed the first test also passed the second one?

(d) If P(A) = 10%, P(B) = 45%, and $P(A \cup B) = 50\%$, find $P(A \mid B)$. *Hint: Look at Special Sum Rule from Before!*

10. Ms. Ganji is checking for homework completion. Each student has a 60% chance of having completed their homework. Ms. Ganji selects two students at random one after another for a homework check. What is the probability that:

    (a) both students have completed their homework?

    (b) neither student has completed their homework?

    (c) only one student has completed their homework?

11. * Alice rolls a standard 6-sided die. Bob rolls a second standard 6-sided die. Alice wins if the values shown differ by 1. What is the probability that Alice wins? (Source: 2009 Pascal (Grade 9), #21)

12. * What is the probability of hitting a bullseye on a dartboard if the bullseye has a radius of 1 cm and the board has a radius of 10 cm? Assume every dart hits the board.

    Hint: *Area of a Circle* $= \pi \times r^2$ *where* $\pi \approx 3.14$ *and* $r^2 = radius \times radius$

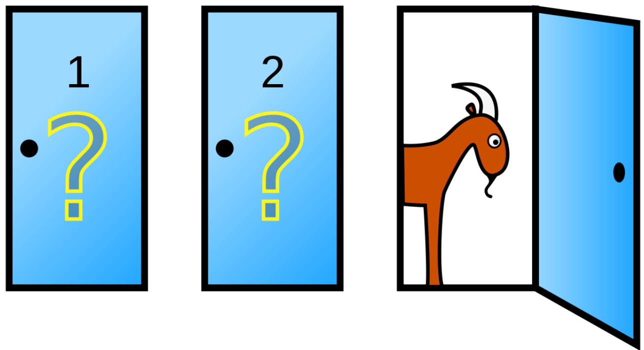
13. * In Canada, 13% of the population plays hockey, basketball and baseball. Additionally, 25% of the population plays basketball and hockey, 16% plays basketball and baseball and 21% plays hockey and baseball. If 28% of the population only play basketball and 15% play only baseball, what percent of the population plays hockey?

    Hint: *Use a Venn Diagram to help you visualize.*

14. * **The Monty Hall Problem**

This is a famous math problem that deals with probability.

You are on a gameshow where you're asked to pick one of three closed doors. Behind two of the three doors there are goats. But behind one of them, there's a brand new car.



(a) What is the probability of winning the car?

(b) You've now picked a door. The gameshow host opens one of the doors you didn't pick and reveals a goat. Now there are two closed doors and one open door with a goat. The host gives you one last chance to change your door. Should you change your mind and pick the other door? Why or why not?

Hint: *Does your probability of winning the car change when the host opens one of the doors? The answer to this question is the key to this problem.*

15. * Three friends are in the park. Bob and Clarise are standing at the same spot and Abe is standing 10 m away. Bob chooses a random direction and walks in this direction until he is 10 m from Clarise. What is the probability that Bob is closer to Abe than Clarise is to Abe? (Source: 2014 Cayley (Grade 10), #23)