

Intermediate Math Circles

February 5, 2020

Contest Prep: Number Theory

Part 1: Prime Factorization

A **prime number** is an integer greater than 1 whose only positive divisors are 1 and itself.

An integer greater than 1 is **composite**, if it is not prime.

How do we check if a number $n > 1$ is prime?

Best test: Check that it does not have any prime divisors between 2 and \sqrt{n} inclusive.

Divisibility Tests:

Divisibility by 2:

A number is divisible by 2 if its last digit is even.

Divisibility by 3:

A number is divisible by 3 if the sum of its digits is divisible by 3.

Ex. 1368: $1 + 3 + 6 + 8 = 18$

Divisibility by 5:

A number is divisible by 5 if its last digit is 0 or 5.

Divisibility by 11:

A number is divisible by 11 if the alternating sum of its digits is divisible by 11.

(An alternating sum is a sum in which the terms alternate sign.)

Ex. 72831: $7 - 2 + 8 - 3 + 1 = 11$

(Although the following divisors are not prime divisors, these tests can be helpful.)

Divisibility by 4:

A number is divisible by 4 if its last two digits are divisible by 4.

Divisibility by 8:

A number is divisible by 8 if its last three digits are divisible by 8.

Divisibility by 9:

A number is divisible by 9 if the sum of its digits is divisible by 9.

615483: $6 + 1 + 5 + 4 + 8 + 3 = 27$

Examples: Are the following numbers prime? If they are composite, express them as a product.

1. 151

Yes, since no prime divisors ≤ 11 .

2. 517

No, since $517 = 11 \times 47$

3. 273

No, since $273 = 3 \times 91$

4. 293

Yes, since no prime divisors ≤ 17

The Fundamental Theorem of Arithmetic: Any integer greater than 1 can be written as product of prime factors.

Examples: Write each number as a product of prime factors.

1. 517

$$517 = 11 \times 47$$

2. 273

$$273 = 3 \times 7 \times 13$$

3. 792

$$792 = 2^3 \times 3^2 \times 11$$

Example: (2018 Pascal #17) Suppose that p and q are two different prime numbers and that $n = p^2q^2$. What is the number of possible values of n with $n < 1000$?

$$n = (pq)^2 < 1000$$

$$pq < \sqrt{1000} \approx 31.6$$

$$2 \times 3 = 6$$

$$2 \times 5 = 10$$

$$2 \times 7 = 14$$

$$2 \times 11 = 22$$

$$2 \times 13 = 26$$

$$2 \times 17 = 34$$

$$3 \times 5 = 15$$

$$3 \times 7 = 21$$

$$3 \times 11 = 33$$

$$5 \times 7 = 35$$

\therefore there are 7 possible values of n .

Example: (2016 Pascal #17) What is the smallest six-digit positive integer that is divisible by each of 10, 11, 12, 13, 14, and 15?

The answer must be divisible by $2 \cdot 5$, 11, $2^2 \cdot 3$, 13, $2 \cdot 7$, and $3 \cdot 5$, so it must be divisible by $2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 60060$.

To get a 6-digit number, multiply 60060 by 2 to get 120120.

Example: (Cayley 1996 #20) Determine the smallest perfect square greater than 4000 that is divisible by 392.

$$\begin{aligned} 392 &= 2(196) \\ &= 2^2(98) \\ &= 2^3(49) \\ &= 2^3 \cdot 7^2 \end{aligned}$$

Any perfect square that is divisible by 392 must be divisible by $2 \cdot 392 = 784$ (so that every prime factor appears an even number of times).

To find the smallest perfect square greater than 4000 that is divisible by 784 we need to solve

$$\begin{aligned} 784n^2 &> 4000 \\ n^2 &> \frac{4000}{784} > 5 \end{aligned}$$

The smallest perfect square greater than 5 is 9.

So the smallest perfect square greater than 4000 which is divisible by 784 is $784 \times 9 = 7056$.

Problem Set 1

1. The prime factorizations of four numbers is given:

$$\begin{aligned} w &= 2^{129} \times 3^{81} \times 5^{128}, \\ x &= 2^{127} \times 3^{81} \times 5^{128}, \\ y &= 2^{126} \times 3^{82} \times 5^{128}, \text{ and} \\ z &= 2^{125} \times 3^{82} \times 5^{129}. \end{aligned}$$

Write w, x, y and z in order from smallest to largest.

Each of w, x, y , and z are divisible by $2^{125} \cdot 3^{81} \cdot 5^{128}$. Let $t = 2^{125} \cdot 3^{81} \cdot 5^{128}$. Therefore, $w = 2^4 t = 16t$, $x = 2^2 t = 4t$, $y = 2 \cdot 3t = 6t$ and $z = 3 \cdot 5t = 15t$. Since $t > 0$, then $x < y < z < w$.

2. The product of 20^{50} and 50^{20} is written as an integer in expanded form. What is the number of zeros at the end of the resulting integer?

$$20^{50} = (2^2 \cdot 5)^{50} = 2^{100} \cdot 5^{50} \text{ and } 50^{20} = (2 \cdot 5^2)^{20} = 2^{20} \cdot 5^{40}.$$

$$\text{So } 20^{50} \cdot 50^{20} = 2^{120} \cdot 5^{90} = (2 \cdot 5)^{90} \cdot 2^{30} = 10^{90} \cdot 2^{30}$$

Thus there will be 90 zeros in the product.

3. The integer 636 405 may be written as the product of three 2-digit positive integers. What are these three integers?

$$636405 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 19 \cdot 29$$

We want to rewrite this as the product of three 2-digit numbers.

Since $3 \cdot 5 \cdot 7 = 105$ which has three digits, and the product of any three of the six prime factors of 636 405 is at least as large as this, then we cannot take the product of three of these prime factors to form a two-digit number.

Thus, we have to combine the six prime factors in pairs.

The prime factor 29 cannot be multiplied by any prime factor larger than 3, since $29 \cdot 3 = 87$ which has two digits, but $29 \cdot 5 = 145$, which has too many digits.

This gives us $636405 = 87 \cdot 5 \cdot 7 \cdot 11 \cdot 19$.

The prime factor 19 can be multiplied by 5 (since $19 \cdot 5 = 95$ which has two digits) but cannot be multiplied by any prime factor larger than 5, since $19 \cdot 7 = 133$, which has too many digits.

This gives us $636405 = 87 \cdot 95 \cdot 7 \cdot 11 = 87 \cdot 95 \cdot 77$.

4. What is the smallest positive integer whose digits have a product of 2700?

In order to find N , which is the smallest possible integer whose digits have a fixed product, we must first find the minimum possible number of digits with this product. (This is because if the integer a has more digits than the integer b , then $a > b$.)

Once we have determined the digits that form N , then the integer N itself is formed by writing the digits in increasing order. (Given a fixed set of digits, the leading digit of N will contribute to the largest place value, and so should be the smallest digit. The next largest place value should get the next smallest digit, and so on.)

Note that the digits of N cannot include 0, or else the product of its digits would be 0.

Also, the digits of N cannot include 1, otherwise we could remove the 1s and obtain an integer with fewer digits (thus, a smaller integer) with the same product of digits.

Since the product of the digits of N is 2700, we find the prime factorization of 2700 to help us determine what the digits are: $2700 = 3^3 \cdot 2^2 \cdot 5^2$

In order for a non-zero digit to have a factor of 5, then the digit must equal 5.

Since 2700 has two factors of 5, then the digits of N includes two 5s.

The remaining digits have a product of $3^3 \cdot 2^2 = 108$.

Therefore, we must try to find a combination of the smallest number of possible digits whose product is 108.

We cannot have a single digit with a product of 108. We also cannot have 2 digits with a product of 108, as the product of 2 digits is at most $9 \cdot 9 = 81$.

We can have a product of 3 digits with a product of 108 (for example, $4 \cdot 9 \cdot 3$ is a possibility).

Therefore, the number N has 5 digits (two 5s and three other digits with a product of 108).

In order for N to be as small as possible, its leading digit (that is, its ten thousands digit) must be as small as possible. Recall that N cannot include the digit 1.

The next smallest possible leading digit is 2. In this case, 2 must be one of the three digits whose product is 108. Thus, the remaining two of these three digits have a product of $108 \div 2 = 54$, and so must be 6 and 9.

Therefore, the digits of N must be 2, 6, 9, 5, 5. The smallest possible number formed by these digits is when the digits are placed in increasing order, and so $N = 25569$.

5. If x and y are integers and $\frac{30!}{36^x 25^y}$ is equal to an integer, what is the maximum possible value of $x + y$?

Since $\frac{30!}{36^x 25^y} = \frac{30!}{6^{2x} 5^{2y}} = \frac{30!}{2^{2x} 3^{2x} 5^{2y}}$, then we need to determine the largest powers of 2, 3, and 5 which divide $30!$.

The product $30!$ includes six factors which are divisible by 5: 5, 10, 15, 20, 25, 30. Each of these contributes one factor of 5 except for 25 which contributes two. So 5^7 is the largest power of 5 which divides $30!$. So $5^{2y} \leq 5^7$ which implies that $2y \leq 7$. Therefore, $y \leq 3$.

The product $30!$ includes ten factors which are divisible by 3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30. Seven of these contributes one factor of 3. Two of them contribute two factors of 3 (9, 18) and one of them contributes three factors of 3 (27). So 3^{14} is the largest power of 3 which divides $30!$. So $3^{2x} \leq 3^{14}$ which implies that $2x \leq 14$. Therefore, $x \leq 7$.

$30!$ includes 15 factors which are divisible by 2. There are already more than 14 factors of 2 so the number of factors of 3 is the limitation which determines the largest value of x . So $x + y \leq 7 + 3 = 10$. The maximum possible value of $x + y$ is 10.

Part 2: Solving Problems Involving Digits

Decimal Expansion

Most often in mathematics we use the **decimal system** of numbers. The digits of a number are multiplied by a power of 10 depending on their position. We say that we are using **base 10**.

For example the decimal expansion of 4327 is $4327 = \underline{4 \times 10^3 + 3 \times 10^2 + 2 \times 10 + 7}$.

In solving some problems that make reference to the digits of numbers we can use a general decimal expansion.

For example, a three digit number can be represented by $\underline{a \times 10^2 + b \times 10 + c}$.

Example: Show that the difference between any three-digit number, in which no digits are 0 or equal, and the number formed by reversing the digits is always a multiple of 99.

$$1 \leq a, b, c \leq 9$$

$$abc = 100a + 10b + c$$

$$cba = 100c + 10b + a$$

$$\begin{aligned} abc - cba &= 100a + 10b + c - (100c + 10b + a) \\ &= 100(a - c) + (c - a) \\ &= 100(a - c) - (a - c) \\ &= 99(a - c) \end{aligned}$$

$a - c$ is an integer \therefore the difference is a multiple of 99.

Example: (Pascal 2003) The people of Evenland never use odd digits. Instead of counting 1, 2, 3, 4, 5, 6, an Evenlander counts 2, 4, 6, 8, 20, 22. What is an Evenlander's version of the integer 1111?

It's like they only have 5 digits to use 0, 2, 4, 6, 8.

$$0 \rightarrow 0$$

$$1 \rightarrow 2$$

$$2 \rightarrow 4$$

$$3 \rightarrow 6$$

$$4 \rightarrow 8$$

$$5 \rightarrow 20$$

$$6 \rightarrow 22$$

$$7 \rightarrow 24$$

$$8 \rightarrow 26$$

$$9 \rightarrow 28$$

We need to use powers of 5:

$$5^0 = 1$$

$$5^1 = 5$$

$$5^2 = 25$$

$$5^3 = 125$$

$$5^4 = 625$$

$$5^5 = 3125$$

$$1111 = \underline{\quad} \times 5^4 + \underline{\quad} \times 5^3 + \underline{\quad} \times 5^2 + \underline{\quad} \times 5^1 + \underline{\quad} \times 5^0$$

$$1111 - 1(625) = 486$$

$$486 - 3(125) = 111$$

$$111 - 4(25) = 11$$

$$11 - 2(5) = 1$$

$$\therefore 1111 = 1 \times 5^4 + 3 \times 5^3 + 4 \times 5^2 + 2 \times 5^1 + 1 \times 5^0$$

\therefore 1111 is represented by 26842

Example: The five-digit number 9T67U, where T and U are single digits, is divisible by 36. Determine all possible values of T and U .

$36 = 2^2 \cdot 3^2$, so the number must be divisible by 4 and 9.

If divisible by 4, then 7U is divisible by 4 $\implies U = 2$ or $U = 6$

If divisible by 9, then sum of the digits is divisible by 9.

$$\text{When } U = 2 : 9 + T + 6 + 7 + 2 = 24 + T$$

$$\text{So } T = 3 \text{ and } (T, U) = (3, 2)$$

$$\text{When } U = 6 : 9 + T + 6 + 7 + 6 = 28 + T$$

$$\text{So } T = 8 \text{ and } (T, U) = (8, 6)$$

Problem Set 2

1. In the multiplication shown, each of P , Q , R , S , and T is a digit.

$$\begin{array}{r} P Q R S T 4 \\ \times 4 \\ \hline 4 P Q R S T \end{array}$$

What is the value of $P + Q + R + S + T$?

Solution 1

We start with the ones digits.

Since $4 \times 4 = 16$, then $T = 6$ and we carry 1 to the tens column.

Looking at the tens column, since $4 \times 6 + 1 = 25$, then $S = 5$ and we carry 2 to the hundreds column.

Looking at the hundreds column, since $4 \times 5 + 2 = 22$, then $R = 2$ and we carry 2 to the thousands column.

Looking at the thousands column, since $4 \times 2 + 2 = 10$, then $Q = 0$ and we carry 1 to the ten thousands column.

Looking at the ten thousands column, since $4 \times 0 + 1 = 1$, then $P = 1$ and we carry 0 to the hundred thousands column.

Looking at the hundred thousands column, $4 \times 1 + 0 = 4$, as expected.

This gives the following completed multiplication:

$$\begin{array}{r} 102564 \\ \times 4 \\ \hline 410256 \end{array}$$

Finally, $P + Q + R + S + T = 1 + 0 + 2 + 5 + 6 = 14$.

Solution 2

Let x be the five-digit integer with digits $PQRST$.

This means that $PQRST0 = 10x$ and so $PQRST4 = 10x + 4$.

Also, $4PQRST = 400\,000 + PQRST = 400\,000 + x$.

From the given multiplication, $4(10x + 4) = 400\,000 + x$ which gives

$40x + 16 = 400\,000 + x$ or $39x = 399\,984$.

Thus, $x = \frac{399\,984}{39} = 10\,256$.

Since $PQRST = 10\,256$, then $P + Q + R + S + T = 1 + 0 + 2 + 5 + 6 = 14$.

2. How many two-digit positive integers are increased by 11 when the order of the digits is reversed?

Let the two-digit integer be of the form ab , where a and b are single digits.

Then the number can be written as $10a + b$ and if the digits are reversed we have $10b + a$.

We are looking for values of a and b such that $10b + a - (10a + b) = 11$.

Collecting like terms this equation becomes $-9a + 9b = 11$.

If we solve for b we get $b = a + \frac{11}{9}$.

But this equation has no solution since a and b are integers. Thus there are no two-digit positive integers that are increased by 11 when the order of their digits is reversed.

3. The three-digit number $2A4$ is added to 329 and gives $5B3$. If $5B3$ is divisible by 3 , then what is the largest possible value of A ?

Since $5B3$ is divisible by 3 , thus $5 + B + 3 = 8 + B$ must be divisible by 3 . Thus the possible values of B are $1, 4$ and 7 .

$$\begin{array}{r} 2 \ A \ 4 \\ + \ 3 \ 2 \ 9 \\ \hline 5 \ B \ 3 \end{array}$$

Since there is a 1 to carry we get that $A + 3 = B$ from the second column. Since nothing is carried into the third column and since we want to maximize A , the largest value of A is 4 and occurs when $B = 7$.

4. A four-digit number which is a perfect square is created by writing Anne's age in years followed by Tom's age in years. Similarly, in 31 years, their ages in the same order will again form a four-digit perfect square. Determine the present ages of Anne and Tom.

Their ages must be two-digit numbers, for if either is one-digit then the other must be three digits, and when 31 is added to the one-digit age the result is a two-digit number. So the adjusted sum would be five digits.

Let a be Anne's age and let t be Tom's age, where a and t are two-digit numbers.

So $100a + t$ is a perfect square, which means that we can write $100a + t = k^2$ for some integer $k > 0$.

Similarly in 31 years from now their ages must be the two-digit numbers $a + 31$ and $t + 31$ and we have that $100(a + 31) + t + 31 = m^2$ for some integer $m > 0$.

Simplifying we obtain that $100a + t + 3131 = m^2$.

We can substitute $k^2 = 100a + t$ to obtain $k^2 + 3131 = m^2$.

Thus $3131 = m^2 - k^2 = (m - k)(m + k)$ using the difference of squares formula.

The prime factorization of 3131 is $31 \cdot 101$. Therefore the factors must be $m - k = 31$ and $m + k = 101$.

If we add these two equations together we get $2m = 132$ and so $m = 66$. Therefore $k = 101 - 66 = 35$.

$35^2 = 1225$ and so Anne's age is 12 and Tom's age is 25 .