Intermediate Math Circles Notes
Wednesday February 26, 2020
Patrick Naylor

This session will be an introduction to *mathematical games*. They seem a bit abstract, but the ideas have far-reaching applications to game theory, economics, and life! To start, we'll look at *combinatorial games*. These games are characterized by the following properties:

- Two players **take turns** (most games!);
- **No luck or chance** is involved (this rules out many card games); and
- Players have **perfect information**: in other words, any information about the game is available to both players (this also rules out many card games).

If we say that the game is *impartial*, we mean the following property:

- **Either player may make any move** (this rules out chess or checkers);

What kinds of games satisfy these properties? As it turns out, quite a few! The first one we'll look at is from the popular TV show *Survivor*.

1. **Thai 21**

   There are 21 flags. Players take turns removing 1, 2, or 3 flags. The player who takes the last flag wins.

   Solution:

   In this game, you always want to go first! This is because if you go first, no matter what your opponent does, you can arrange that the number of flags remaining is a multiple of 4. For instance, if there are 4 flags left and it's your opponents turn, *no matter what they do*, you can win! Similarly, if there are 8 flags left and it's your opponents turn, you can make sure that on their next turn, there are 4 flags left (and so on). Since there are 21 flags to start, if you go first you will always be able to win.

   If the player who takes the last flag loses, this is almost the same game! The player who takes the second last flag will win, since the other player will have no choice but to take the last one. In this case, the second player will always be able to win, since they can always make sure the number of flags is 1 plus a multiple of 4 (think about what happens when there are 5 flags left).

2. **Pick and Split**

   There are two piles of stones. One has 10 stones, and the other has 13 stones. On each players turn, they must discard one pile of stones and split the other pile into two piles. Every pile must have at least one stone. The first player unable to make a legal move loses. In other words, the game ends when there are two piles of one stone each.

   Solution:

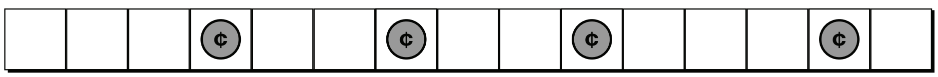
   In this game, you want to go first! This strategy is a bit tricky to come up with, but easy once you know what it is. Discard the pile of 13 stones, and divide the pile of 10 into two piles with an *odd* number of stones in each, say 7 and 3. No matter what the second player does, they must throw away one of these piles, and divide the other into two piles: one will be even, and one will odd. Once again, *no matter what they choose*, you can leave them with two piles with an odd number of stones. Eventually, they will be left with two piles of 1 stone each, and you will win.

You should convince yourself that the first player can always win as long as one of the starting piles has an even number of stones. Otherwise, the second player can always win.

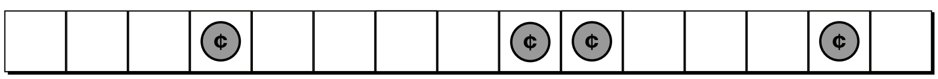
In both these games, the first player can always win. If they make their moves carefully enough, they won't lose. This isn't an accident! For any impartial combinatorial game, one of the players always has a *winning strategy*.

3. **Sliding Pennies**

The game begins with four pennies placed on a 1×15 grid, just like the picture below.



Players take turns moving one penny to the right, but may not slide pennies past each other. For instance, a legal move for Player 1 might be the following.



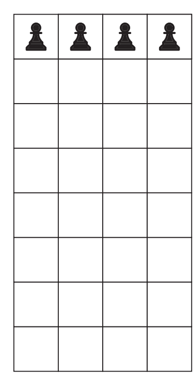
The first player unable to make a move loses. In other words, the game ends when the pennies are stacked all the way on the right side of the grid.

Solution:

We'll discuss this problem next week! Stay tuned.

Hint: There is a common theme to the problems in this worksheet. This may help you discover a winning strategy.

4. **Half a Chessboard**



There is a 4×8 (vertical) chessboard with four pawns in the top row. On each turn, players can move $1, 2, 3$, or all 4 pawns down one square, until they reach the bottom row. The player who cannot make a move loses.

- Do you want to go first or second?
- What if the pawns are allowed to start in any position (one in each column)?
- What if there are 5 pawns on a 5×8 chessboard? What if there are 4 pawns on an $4 \times n$ chessboard?

Solution:

Like **Sliding Pennies**, the solution to this game is about symmetry. Think about labelling the rows of the chessboard, so that the bottom is row 0, and the top is row 7. Once again, you want to go first. Move all the pawns from row 7 down to row 6. Note that no matter what your opponent does, you can always move some of the pawns so that once again, they all lie on an even numbered row. Eventually, you will move every pawn onto the bottom row (row 0), and your opponent won't be able to make a move. You win!

What are some *similarities* between the winning strategies for these games? What are some *differences*?

More Problems!

If you liked those games, see if you can figure out winning strategies for these ones. Some of them are tricky! Solutions are not provided.

5. **Erase from 13**

   A chalkboard has the numbers 1,2,3,...,13 written on it. Two players take turns erasing a number from the board, until two numbers remain. The first player wins if the sum of the last two numbers is a multiple of 3. Otherwise, the second player wins.

   What if we start with the numbers $1, 2, 3, ..., 2020$?

6. **Generalized Thai 21**

   There are n flags. On their turn, a player can remove 1, 2, 3,..., or k flags. The player who takes the last flag wins.

7. **Even More Thai 21**

   There are 21 flags. On their turn, a player can remove 1, 2, or 4 flags. The player who takes the last flag wins.

   There are n flags. On their turn, a player can remove 1, 3, or 5 flags. The player who takes the last flag wins.

8. **Sliding Nickels**

   Just like **Sliding Pennies**, the games begins with 5 (not 4!) nickels placed on a 1×15 grid. Players take turns sliding one nickel to the right, but may not slide nickels past each other. The first player unable to make a legal move loses.

9. **The Subtracting Game**

   Similar to **Thai 21**, this game starts with 44 flags. Player 1 can remove any number of flags, but must leave at least one. Thereafter, players may remove *at most* as many as the previous player did. The player who takes the last flag wins.

10. **Yet Another Flag Game**

    Like **The Subtracting Game**, this game starts with 44 flags. Player 1 can remove any number of flags, but must leave at least one. Thereafter, players may remove *up to twice as many* as the previous player did. The player who takes the last flag wins.