Intermediate Math Circles Problem Set

Wednesday March 4, 2020

Patrick Naylor

We looked at the following games this week, and tried to determine a winning strategy.

1. **The Calendar Game**

Players take turns writing down dates. The first player must begin by writing down January 1. After this, the next player takes the previous date and may increase either the month or the day, but *not both*. For example, the second player could choose January 12, or May 1, but not Feburary 2. The player who writes down December 31 wins.

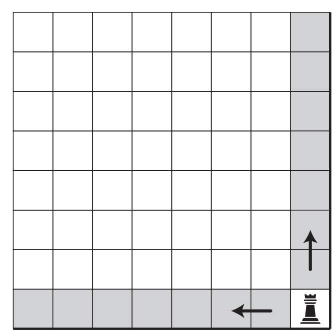
Solution:

Working backwards, notice that if you write down November 30, you will win. Indeed, no matter what your opponent chooses, you will be able to write down December 31. Similarly, if you write down October 29, no matter what your opponent chooses, you will be able to write down either November 30, or December 31. Continuing in this fashion, the same logic applies to September 28, August 27, and so on. In particular, if you go first and write down January 20, you will be able to win!

2. **A Rook on a Chessboard**

A rook is placed on the bottom right square of an 8×8 chessboard. On each players turn, they move it any number of spaces to the left, or any number of spaces up (never to the right or down). The player who moves the rook to the top left square wins.

- Do you want to go first or second?
- What if the game was played on an $n \times n$ chessboard?



Solution:

The solution to this game is to go second, and no matter what your opponent does, move back to the diagonal squares. Since you can't move the rook to a new diagonal square without first moving off, you will eventually win!

These games are actually the same. If we draw a labelled chessboard with months on one side and dates on the other, the squares we want to move to are exactly the diagonal ones (November 30, October 29,...). The only difference between these games is whether or not you start on the diagonal.

Sometimes, drawing a picture or a schematic makes understanding these games easier. These games are the same, but **A Rook on a Chessboard** seems much simpler.

Winning and losing positions are a useful idea that will make our study of combinatorial games easier.

A **winning position** is a position with the property that if it's your turn, you can guarantee that you can win.

A **losing position** is a position that is not a winning position. In other words, if it's your turn, you can't guarantee you'll be able to win.

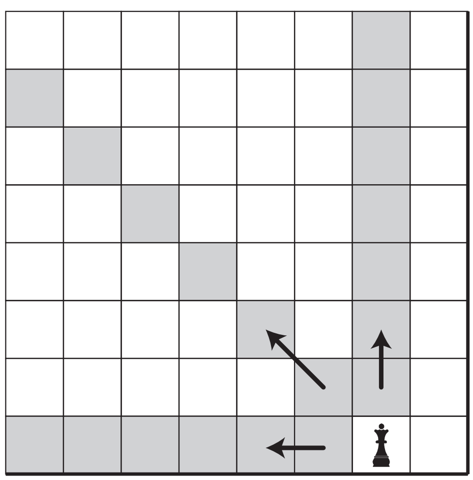
These positions actually have a lot to do with each other. Notice: there must be *at least one* move from a winning position to some losing position. In other words, there's some (good) move you can make that leaves your opponent in a losing position. Otherwise, your opponent would be able to win! Conversely, every move from a losing position must be to a winning position, or otherwise you wouldn't be able to guarantee you can win.

This is illustrated in the chessboard below: every move starting on a square labelled L ends on a square labelled W, and every square labelled W has a move to *some* square labelled L (or the top left corner).

3. **The Left Handed Queen**

   A queen is placed near the bottom right square of an 8×8 chessboard. On each players turn, they can move it any number of spaces to the left, diagonally up and to the left, or up. The player who moves the queen to the top left square wins.

   - Do you want to go first or second?
   - What if the queen starts in a different square? Can you still tell if you want to go first or second?
   - Label all the winning and losing positions on the empty chessboard below. Can you find a pattern?



	W	W	W	W	W	W	W
W	W	L	W	W	W	W	W
W	L	W	W	W	W	W	W
W	W	W	W	W	L	W	W
W	W	W	W	W	W	W	L
W	W	W	L	W	W	W	W
W	W	W	W	W	W	W	W
W	W	W	W	L	W	W	W

Solution:

Working backwards, we can label the chessboard with winning and losing positions. In particular, the starting position given is a winning position. If we add labels to the chessboard starting at zero, then the (top) losing positions have coordinates:

$$\{(1,2),(3,5),(4,7),(6,10),(8,13),(9,15),(11,18),(12,21),...\}$$

What is this sequence?

4. **Wythoff's Game**

The game begins with a pile of 7 coins, and a pile of 11 coins. On their turn, each player may take any number of coins from either pile, or *the same number* of coins from both. The player who takes the last coin wins.

- Do you want to go first or second?
- Is this is the same as **The Left Handed Queen**? How can you tell?
- What if we start with piles of 9 and 14 coins?

Solution:

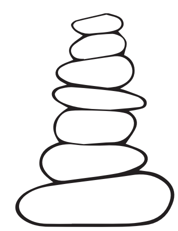
The same way that **The Calendar Game** is the same as **A Rook on a Chessboard**, this game is the same as **The Left Handed Queen**! Label the sides of a chessboard like before, and see what happens.

Sometimes (in fact, frequently), combinatorial games don't have winning strategies that are easy to describe. A *game tree* is a useful way to understand combinatorial games. The classic game **Nim** is a good example that we'll try to understand this way.

5. **Easy Nim**

The game begins with two piles of stones. One has 5 and the other has 7. On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone wins.

- Do you want to go first or second? What is the winning strategy?
- Have we seen this game before?



Solution:

Easy Nim isn't really **Nim**. This is just another example of a diagonal game! The winning strategy here is to go first, and make the number of stones in both piles equal. In other words, **Nim** with two piles really is easy.

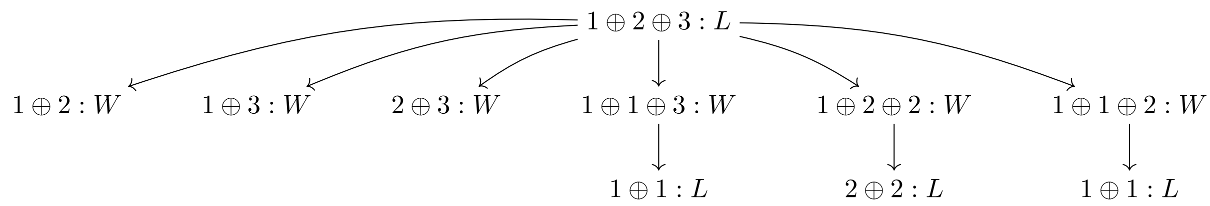
6. **Actual Nim**

The game begins with five piles of stones. There are 1, 2, 3, 4, and 5 stones in each pile. On their turn, each player may take *any* number of stones from one pile. The player who takes the last stone wins.

- What if we only start with 1, 2, and 3 stones ($1 \oplus 2 \oplus 3$)? Is this a winning position?
- Why is this different from **Easy Nim**?
- Is $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5$ a winning position?
- What if we started with 10 piles? With n piles?
- What is the winning strategy? Is it easy to describe?

Solution:

The game tree for $1 \oplus 2 \oplus 3$ is drawn below. Note that from our experience with **Easy Nim**, we know that any position with two piles that are equal is a losing position. Hence, any position with a move to such a position is a winning position!



Since all moves from $1 \oplus 2 \oplus 3$ are to winning positions, it must be a losing position. Stay tuned- we'll talk about the full solution to **Nim** next week.

More Problems!

If you liked those games, see if you can figure out winning strategies for these ones. Some of them are tricky!

7. **Erase from 13**

   A chalkboard has the numbers $1, 2, 3, ..., 13$ written on it. Two players take turns erasing a number from the board, until two numbers remain. The first player wins if the sum of the last two numbers is a multiple of 3. Otherwise, the second player wins.

   What if we start with the numbers $1, 2, 3, ..., 2020$?

8. **A Knight on a Chessboard Game?**

   The game begins with a knight placed on an 8×8 chessboard. Players take turns moving the knight in the usual L-shaped moves. The player who moves it to the top left corner wins.

   - Why is this *not* a combinatorial game?

9. **A Real Knight on a Chessboard Game**

   There is a way to make the above game into a real combinatorial game. The first player *chooses* a starting position for a knight anywhere on an 8×8 chessboard. Afterwards, players take turns moving the knight to a position *that has not been visited before* (in otherwords, no repeated positions are allowed). The player who has no more available moves loses.

   - Why is this a combinatorial game?
   - This is a hard problem! Show that the second player has a winning strategy if the board has size 2×4. If you can do this, show that the second player has a winning strategy if the game is played on a 4×4 or 5×5 chessboard.
   - Can a knight placed on a chessboard visit every square exactly once?
   - If you can do this, prove that the second player has a winning strategy on a regular 8×8 chessboard.

10. **Circular Nim**

    The game begins with 10 stones placed in a circle. Players take turns removing up to three *consecutive* stones. The player who takes the last stone wins.