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Grade 7/8 Math Circles

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Infinite Series

Introduction

When we add infinitely many numbers, it is called an **infinite series**.

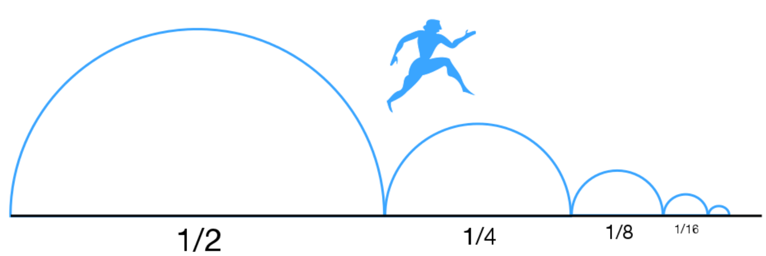
What is $1 + 1 + 1 + 1 + 1 + 1 + 1 + ... = ?$

What is $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... = ?$

What is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ... = ?$

Zeno's Paradoxes of Motion

Suppose I want to walk from one end of this room to the other. First I have to walk halfway there, then I have to walk through half of the leftover distance, and so on. This keeps repeating over and over, so I can never reach the other end of the room. This problem was posed by the ancient Greek philosopher Zeno in the 5th century BCE.



Retrieved from: `https://commons.wikimedia.org/wiki/File:Zeno_Dichotomy_Paradox.png`

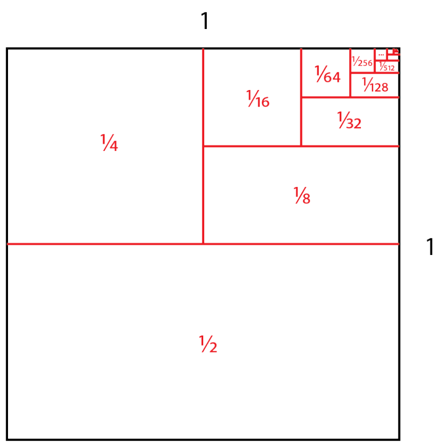
Clearly, I can walk across this room, so where is the flaw in Zeno's argument? It took another 2000 years and the invention of calculus for this question to be answered.

Convergent Series

The main assumption made in Zeno's paradox seems intuitive: the sum of infinitely many numbers is infinite. However, this is not actually true!

Consider the series from before: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ... = ?$

We can create a geometric representation of this series by filling in the following diagram:



Still don't believe me? Let's look at an arithmetic proof together. Feel free to copy it down in the space below:

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$$

$$\frac{1}{2} \times S = \frac{1}{2} \times (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...)$$

$$\frac{1}{2} \times S = (\frac{1}{2} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{4}) + (\frac{1}{2} \times \frac{1}{8}) + (\frac{1}{2} \times \frac{1}{16}) + ...$$

$$\frac{1}{2} \times S = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$$

$$\frac{1}{2} \times S = S - \frac{1}{2}$$

$$2 \times \frac{1}{2} \times S = 2 \times (S - \frac{1}{2})$$

$$S = 2S - 1$$

$$S + 1 - S = 2S - 1 + 1 - S$$

$$1 = S$$

An infinite series is called **convergent** if the sum of the series is equal to a finite number (as opposed to going off to infinity).

Divergent Series

The opposite of a convergent series is a **divergent** series. The example from the introduction, 1 + 1 + 1 + ..., is a divergent series because its sum increases without bound.

Consider this series, called the **harmonic series**:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$$

Can we determine what this sum is?

We actually cannot. This series is divergent. This fact was first proved by French philosopher Nicole Oresme in the 14th century. Let's look at his proof together. Again, feel free to copy it down in the space below:

This is our starting series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + ...$$

Let's make a smaller series. To do so, we will make some of the individual fractions smaller by increasing the denominator of each fraction to the nearest power of 2. So 1 becomes $\frac{1}{2}$, $\frac{1}{3}$ becomes $\frac{1}{4}$, and $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$ each become $\frac{1}{8}$:

$$\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + ...$$

$$= \frac{1}{2} + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + ...$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ...$$

This sum increases without bound, so it is clearly a divergent series. If this series goes to infinity, then our original series, which is larger than this one, must do so as well.

Infinite Geometric Series

In an **infinite geometric series**, each term is formed by multiplying the previous term by some common ratio. For example:

$$1 + 3 + 9 + 27 + 81 + 243 + ...$$

is a geometric series that starts at ___1___ and the common ratio is ___3___.

$$5 + 25 + 125 + 625 + 3125 + ...$$

is a geometric series that starts at ___5___ and the common ratio is ___5___.

$$3 + 15 + 75 + 375 + 1875 + ...$$

is a geometric series that starts at ___3___ and the common ratio is ___5___.

Let's call the number we start at "a", and the common ratio "r".

Exercise: Write out the first 5 terms in the following series:

1. a = 1 and r = $\frac{1}{2}$:

   $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$ converges

   Does this series converge or diverge? (hint: we have seen this series before!)

2. a = 1, r = 1:

   $1 + 1 + 1 + 1 + 1 + ...$ diverges

   Does this series converge or diverge?

3. a = 2, r = 3:

   $2 + 6 + 18 + 54 + 162 + ...$ diverges

   Does this series converge or diverge?

Can you see a pattern for whether these series converge or diverge?

It turns out, for a geometric series, there is a simple rule. If $|r| \geq 1$, the series diverges. Otherwise, it converges. Furthermore, if it converges, the sum of the series equals $\frac{a}{1-r}$. Let's prove this formula together (hint: it's a generalization of our proof from page 2!):

$$S = a + ar + ar^2 + ar^3 + ar^4 + ...$$

$$S \times r = r \times (a + ar + ar^2 + ar^3 + ar^4 + ...)$$

$$S \times r = (r \times a) + (r \times ar) + (r \times ar^2) + (r \times ar^3) + (r \times ar^4) + ...$$

$$S \times r = ar + ar^2 + ar^3 + ar^4 + ar^5 + ...$$

$$S \times r = S - a$$

$$S \times r - S \times r + a = S - a - S \times r + a$$

$$a = S - S \times r$$

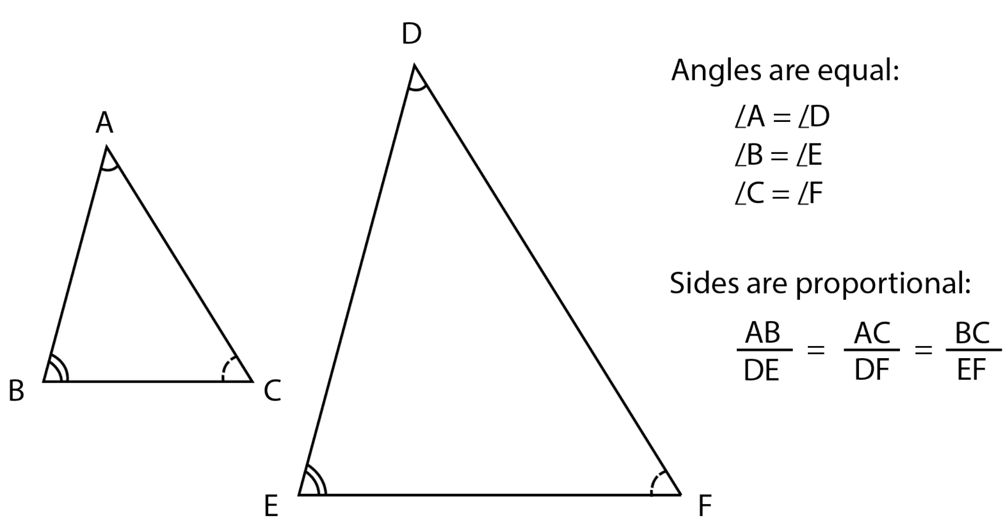
$$a = S \times (1 - r)$$

$$\frac{a}{1-r} = S$$

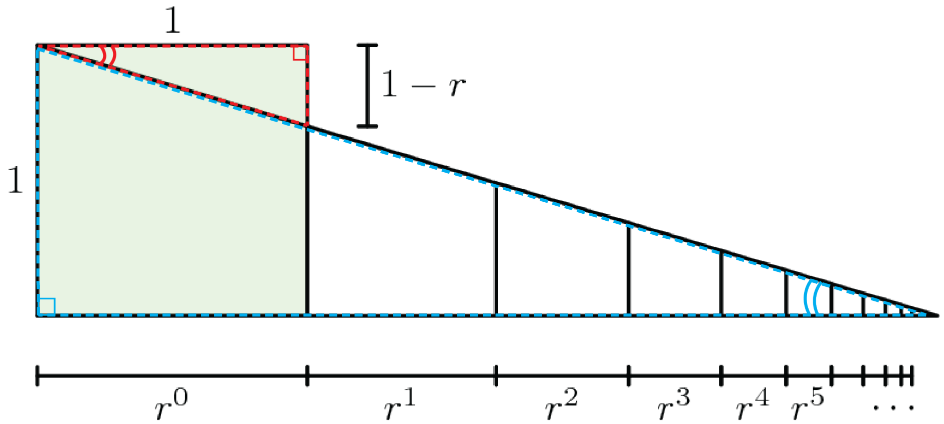
We can also use a visual proof. First, let's do a quick review of similar triangles.

Similar Triangles

Similar triangles have the same shape, but not necessarily the same size. Their corresponsing angles are equal, and their corresponding side lengths are **proportional**.



Now we can look at the proof. Feel free to make notes on the proof below the image.



Retrieved from: `https://artofproblemsolving.com/wiki/index.php/Proofs_without_words`

In the diagram above, the indicated angles are equal because of the Z-rule (from Lesson 1). These are right-angled triangles, so by the interior angles of a triangle rule, their third angles must also be equal. Therefore, the two triangles are similar.

Since they are similar, their side lengths are proportional, so:

$$\frac{r^0 + r^1 + r^2 + ...}{1} = \frac{1}{1-r}$$

$$1 + r + r^2 + ... = \frac{1}{1-r}$$

This formula uses $a = 1$. If $a \neq 1$, just multiply both sides of the equation by a to get:

$$a + ar + ar^2 + ... = \frac{a}{1-r}$$

Exercise: Determine if the series is convergent or divergent, and if it is convergent, state the sum of the series.

1. $1 + 2 + 4 + 8 + 16 + ... =$ N/A (divergent)
   a: 1
   r: 2

2. $4 + 8 + 16 + 32 + 64 + ... =$ N/A (divergent)
   a: 4
   r: 2

3. $20 + 10 + 5 + \frac{5}{2} + \frac{5}{4} + ... = 40$ (convergent)
   a: 20
   r: $\frac{1}{2}$

4. $120 + 60 + 30 + 15 + \frac{15}{2} + ... = 240$ (convergent)
   a: 120
   r: $\frac{1}{2}$

5. $\frac{1}{2} + \frac{5}{2} + \frac{25}{2} + \frac{125}{2} + \frac{625}{2} + ... =$ N/A (divergent)
   a: $\frac{1}{2}$
   r: 5

6. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + ... = \frac{3}{2}$ (convergent)
   a: 1
   r: $\frac{1}{3}$

7. $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + ... = \frac{10}{9}$ (convergent)
   a: 1
   r: $\frac{1}{10}$
   Can you write out this series in decimal form?
   $1 + 0.1 + 0.01 + 0.001 + 0.0001 + ... = 1.111...$

Take another look at that last example. We have just discovered a trick to turn a repeating decimal into a fraction! Let's do an example together:

Example: Write 0.5555555... as a fraction:

$$\begin{aligned} 0.5555555... &= 0.5 + 0.05 + 0.005 + 0.0005 + ... \\ &= \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \frac{5}{10000} + ... \\ &= \frac{\frac{5}{10}}{1 - \frac{1}{10}} \\ &= \frac{5}{9} \end{aligned}$$

Exercise: Express the following repeating decimals as fractions:

1. 0.333333...

$$\begin{aligned} 0.333333... &= 0.3 + 0.03 + 0.003 + 0.0003 + ... \\ &= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + ... \\ &= \frac{\frac{3}{10}}{1 - \frac{1}{10}} \\ &= \frac{1}{3} \end{aligned}$$

2. 1.222222...

$$\begin{aligned} 1.222222... &= 1 + 0.2 + 0.02 + 0.002 + 0.0002 + ... \\ &= 1 + \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + ... \\ &= 1 + \frac{\frac{2}{10}}{1 - \frac{1}{10}} \\ &= \frac{11}{9} \end{aligned}$$

Remember the Koch Snowflake from our last lecture? Let's use an infinite geometric series to calculate its area:

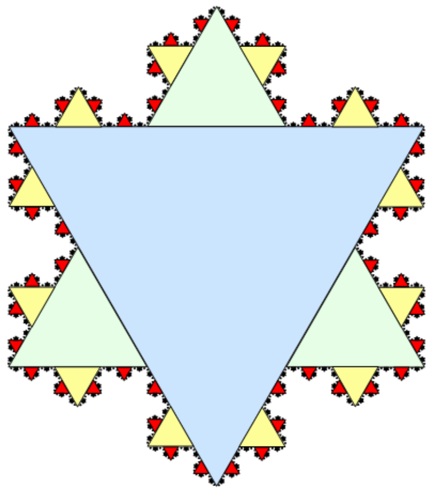


Image by Jim Belk, Retrieved from:
https://commons.wikimedia.org/wiki/File:Koch_Snowflake_Triangles.png

The Koch Snowflake is made up of an infinite number of equilateral triangles. If a is the side length of the equilateral triangle, its area is:

$$Area = \frac{\sqrt{3}}{4}a^2$$

To create the snowflake, recall that we added triangles that had $\frac{1}{3}$ the side length of the previous triangle. Using the formula, what is the area of a green triangle compared to a blue one? A yellow triangle compared to a green one?

If the side length of a blue triangle is a, then the side length of a green triangle is $\frac{a}{3}$. Therefore the area of a green triangle is $\frac{\sqrt{3}}{4}(\frac{a}{3})^2 = \frac{\sqrt{3}}{4}\frac{a}{3}\frac{a}{3} = \frac{\sqrt{3}}{4}\frac{a^2}{9} = \frac{\text{area of a blue triangle}}{9}$. Similarly, the area of a yellow triangle is $\frac{1}{9}$ the size of a green triangle, and so on for the other triangles.

How many green triangles are there? 3 Yellow? 12 Red? 48

Each green triangle is $\frac{1}{9}$ the size of a blue one, and each yellow triangle is $\frac{1}{9}$ the size of a green one, and each red triangle is $\frac{1}{9}$ the size of a yellow one, and so on. Additionally, the Snowflake has 1 blue triangle, 3 green triangles, 12 yellow triangles, and 48 red triangles, and so on.

Putting this information all together, we get that the area of the Koch Snowflake, assuming a blue triangle has area 1, is:

$$1 + 3\left(\frac{1}{9}\right) + 12\left(\frac{1}{9}\right)^2 + 48\left(\frac{1}{9}\right)^3 + ...$$

If we don't include the initial 1, we have a geometric series!

a $= 3\left(\frac{1}{9}\right) = \frac{1}{3}$

r $= 4\left(\frac{1}{9}\right) = \frac{4}{9}$

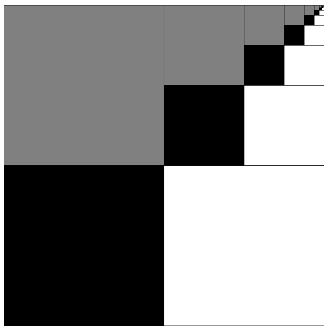
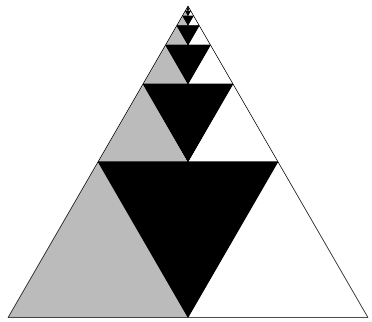
$\frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{4}{9}} = \frac{\frac{1}{3}}{\frac{5}{9}} = \frac{1}{3} \times \frac{9}{5} = \frac{3}{5}$

So the area of the Koch Snowflake is $1 + \frac{3}{5} = \frac{8}{5}$

Problem Set

Questions marked with an asterisk (*) are challenge questions

1. Based on the geometric proofs, can you tell what series we are summing, and what the sum is equal to?

   (a)  or 

   (These two diagrams show the same series!)

   Retrieved from: https://en.wikipedia.org/wiki/File:Geometric_series_14_square2.svg and https://commons.wikimedia.org/wiki/File:Geometric_series_triangle.svg

   $$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + ... = \frac{1}{3}$$

   (b) 

   $$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + ... = \frac{1}{2}$$

2. Write out the first 4 terms of the following geometric series based on the given values of a and r and identify whether each converges or diverges:

   (a) a = 45, r = 10

   45 + 450 + 4500 + 45000 + ... (diverges)

   (b) a = 1, r = $\frac{1}{5}$

   $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + ...$ (converges)

   (c) a = 15, r = $\frac{1}{3}$

   $15 + 5 + \frac{5}{3} + \frac{5}{9} + ...$ (converges)

   (d) a = $\frac{1}{5}$, r = 2

   $\frac{1}{5} + \frac{2}{5} + \frac{4}{5} + \frac{8}{5} + ...$ (diverges)

   (e) a = 1, r = $\frac{1}{100}$

   $1 + \frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + ...$ (converges)

   (f) a = 3, r = 1

   3 + 3 + 3 + 3 + ... (diverges)

3. Find the value of a and r for the following geometric series. Do they converge or diverge?

   (a) 3 + 30 + 300 + 3000 + 30000 + ...

   a = 3, r = 10, diverges

   (b) $4 + 4 + 4 + 4 + 4 + ...$

   a = 4, r = 1, diverges

   (c) $25 + 5 + 1 + \frac{1}{5} + \frac{1}{25} + ...$

   a = 25, r = $\frac{1}{5}$, converges

   (d) $5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \frac{5}{81} + ...$

   a = 5, r = $\frac{1}{3}$, converges

   (e) $\frac{1}{9} + \frac{1}{3} + 1 + 3 + 9 + ...$

   a = $\frac{1}{9}$, r = 3, diverges

   (f) $300 + 50 + \frac{25}{3} + \frac{25}{18} + \frac{25}{108} + ...$

   a = 300, r = $\frac{1}{6}$, converges

4. Find the sum of the convergent geometric series from the previous two questions.

   2. b) $\frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$

   2. c) $\frac{15}{1-\frac{1}{3}} = \frac{15}{\frac{2}{3}} = \frac{45}{2}$

   2. e) $\frac{1}{1-\frac{1}{100}} = \frac{1}{\frac{99}{100}} = \frac{100}{99}$

   3. c) $\frac{25}{1-\frac{1}{5}} = \frac{25}{\frac{4}{5}} = \frac{125}{4}$

   3. d) $\frac{5}{1-\frac{1}{3}} = \frac{5}{\frac{2}{3}} = \frac{15}{2}$

   3. f) $\frac{300}{1-\frac{1}{6}} = \frac{300}{\frac{5}{6}} = 360$

5. Write the following repeated decimals as fractions:

(a) 0.8888...

$$\begin{aligned} 0.8888... &= 0.8 + 0.08 + 0.008 + ... \\ &= \frac{8}{10} + \frac{8}{100} + \frac{8}{1000} + ... \\ &= \frac{\frac{8}{10}}{1 - \frac{1}{10}} \\ &= \frac{\frac{4}{5}}{\frac{9}{10}} \\ &= \frac{4}{5} \times \frac{10}{9} \\ &= \frac{8}{9} \end{aligned}$$

(b) 1.4444...

$$\begin{aligned} 1.444... &= 1 + 0.4 + 0.04 + 0.004 + ... \\ &= 1 + \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + ... \\ &= 1 + \frac{\frac{4}{10}}{1 - \frac{1}{10}} \\ &= 1 + \frac{\frac{4}{10}}{\frac{9}{10}} \\ &= 1 + \frac{4}{10} \times \frac{10}{9} \\ &= \frac{13}{9} \end{aligned}$$

(c) 2.11111...

$$\begin{aligned} 2.11111... &= 2 + 0.1 + 0.01 + 0.001 + ... \\ &= 2 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + ... \\ &= 2 + \frac{\frac{1}{10}}{1 - \frac{1}{10}} \\ &= 2 + \frac{\frac{1}{10}}{\frac{9}{10}} \\ &= 2 + \frac{1}{10} \times \frac{10}{9} \\ &= \frac{19}{9} \end{aligned}$$

(d) 3.3333...

$$\begin{aligned}
3.3333... &= 3 + 0.3 + 0.03 + 0.003 + ... \\
&= 3 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ... \\
&= \frac{3}{1 - \frac{1}{10}} \\
&= \frac{3}{\frac{9}{10}} \\
&= 3 \times \frac{10}{9} \\
&= \frac{10}{3}
\end{aligned}$$

(e) * 0.454545...

$$\begin{aligned}
0.454545... &= 0.45 + 0.0045 + 0.000045 + ... \\
&= \frac{45}{100} + \frac{45}{10000} + \frac{45}{1000000} + ... \\
&= \frac{\frac{45}{100}}{1 - \frac{1}{100}} \\
&= \frac{\frac{9}{20}}{\frac{99}{100}} \\
&= \frac{9}{20} \times \frac{100}{99} \\
&= \frac{5}{11}
\end{aligned}$$

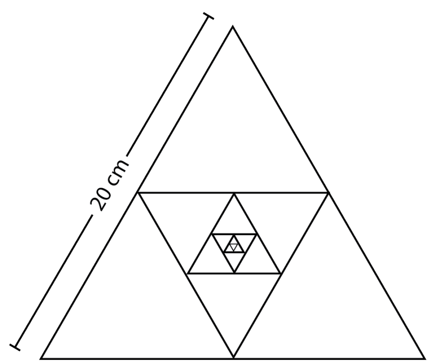
(f) * 0.123123123...

$$\begin{aligned}
0.123123123... &= 0.123 + 0.000123 + 0.000000123 + ... \\
&= \frac{123}{1000} + \frac{123}{1000000} + \frac{123}{1000000000} + ... \\
&= \frac{\frac{123}{1000}}{1 - \frac{1}{1000}} \\
&= \frac{\frac{123}{1000}}{\frac{999}{1000}} \\
&= \frac{123}{1000} \times \frac{1000}{999} \\
&= \frac{41}{333}
\end{aligned}$$

6. At first glance you might disagree, but there are many different proofs for this statement: 0.99999.... = 1. Try proving it yourself using an infinite geometric series.

$$\begin{aligned}0.9999... &= 0.9 + 0.09 + 0.009 + ...\\ &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ...\\ &= \frac{\frac{9}{10}}{1 - \frac{1}{10}}\\ &= \frac{\frac{9}{10}}{\frac{9}{10}}\\ &= 1\end{aligned}$$

7. * An equilateral triangle is drawn with a side length of 20 cm. Another equilateral triangle is formed by joining the midpoints of the sides of the first one. Then a third equilateral triangle is formed by joining the midpoints of the sides of the second, and so on forever. What is the sum of the perimeters of all of the triangles drawn?
(Taken from 1951 AHSME, #9)



The perimeter of the first triangle is 3 sides × 20 cm, the perimeter of the second triangle is 3 sides $\times \frac{20 \text{ cm}}{2}$, and so on. As an infinite geometric series:

$$\begin{aligned}&3 \times 20 + 3 \times \frac{20}{2} + 3 \times \frac{20}{4} + 3 \times \frac{20}{8} ...\\ &= 60 + 30 + 15 + \frac{15}{2} + ...\\ &= \frac{60}{1 - \frac{1}{2}}\\ &= \frac{60}{\frac{1}{2}}\\ &= 120 \text{ cm}\end{aligned}$$

8. * An infinite geometric series has a common ratio of $\frac{3}{4}$ and a sum of 60. What is its first term?

Let the first term be x:

$$
\begin{aligned}
x + \frac{3}{4}x + \frac{9}{16}x + \frac{27}{64}x + ... &= 60 \\
\frac{x}{1 - \frac{3}{4}} &= 60 \\
\frac{x}{\frac{1}{4}} &= 60 \\
\frac{1}{4} \times \frac{x}{\frac{1}{4}} &= \frac{1}{4} \times 60 \\
x &= 15
\end{aligned}
$$

9. * The first term of an infinite geometric series is m, its common ratio is m, and its sum is 10. Find the value of m.

$$
\begin{aligned}
\frac{m}{1-m} &= 10 \\
(1-m) \times \frac{m}{1-m} &= (1-m) \times 10 \\
m &= 10 \times (1-m) \\
m &= 10 - 10m \\
m + 10m &= 10 - 10m + 10m \\
11m &= 10 \\
\frac{11m}{11} &= \frac{10}{11} \\
m &= \frac{10}{11}
\end{aligned}
$$

10. * Find the sum of the following infinite geometric series:

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + ...$$

The first term is 1 and the common ratio is $-\frac{1}{2}$, so using the formula, we get:

$$
\begin{aligned}
&= \frac{1}{1 - (-\frac{1}{2})} \\
&= \frac{1}{1 + \frac{1}{2}} \\
&= \frac{1}{\frac{3}{2}} \\
&= \frac{2}{3}
\end{aligned}
$$

11. * Find the sum of the following infinite geometric series:

$$\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}+\frac{1}{32}-\frac{1}{64}+...$$

(Taken from 1959 AHSME, #49)

Consider the series in groups of three terms. Then we can rewrite it as:

$$\begin{aligned}
&\underbrace{\frac{1}{2}+\frac{1}{4}-\frac{1}{8}}+\underbrace{\frac{1}{16}+\frac{1}{32}-\frac{1}{64}}+...\\
&=\quad\frac{5}{8}\qquad+\qquad\frac{5}{64}\qquad+...\\
&=\frac{\frac{5}{8}}{1-\frac{1}{8}}\\
&=\frac{\frac{5}{8}}{\frac{7}{8}}\\
&=\frac{5}{8}\times\frac{8}{7}\\
&=\frac{5}{7}
\end{aligned}$$

12. * Find the sum of the following infinite geometric series:

$$\frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}+\frac{4}{10^4}+...$$

(Taken from 1962 AHSME, #40)

Let's split up this series and write it in a grid:

$$\frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}+\frac{4}{10^4}+...$$

$$=\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^3}+\frac{1}{10^3}+\frac{1}{10^4}+\frac{1}{10^4}+\frac{1}{10^4}+\frac{1}{10^4}+...$$

$$=\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+... \qquad \leftarrow \text{this is a series (1)}$$

$$+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+... \qquad \leftarrow \text{this is another series! (2)}$$

$$+\frac{1}{10^3}+\frac{1}{10^4}+... \qquad \leftarrow \text{this is a third series! (3)}$$

$$+\frac{1}{10^4}+... \qquad \leftarrow \text{this is a fourth series! (4)}$$

$$\vdots \qquad \leftarrow \text{and so on}$$

This is a series of series! Each row of the grid is a different series.

Series #	a	r	sum
1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{\frac{1}{10}}{1-\frac{1}{10}}=\frac{1}{9}$
2	$\frac{1}{10^2}$	$\frac{1}{10}$	$\frac{\frac{1}{10^2}}{1-\frac{1}{10}}=\frac{1}{90}$
3	$\frac{1}{10^3}$	$\frac{1}{10}$	$\frac{\frac{1}{10^3}}{1-\frac{1}{10}}=\frac{1}{900}$
4	$\frac{1}{10^4}$	$\frac{1}{10}$	$\frac{\frac{1}{10^4}}{1-\frac{1}{10}}=\frac{1}{9000}$
...	...	...	...

$$\therefore \frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}+\frac{4}{10^4}+\dots$$
$$=\text{series 1 }+\text{series 2 }+\text{series 3 }+\text{series 4 }+\dots$$
$$=\frac{1}{9}+\frac{1}{90}+\frac{1}{900}+\frac{1}{9000}+\dots$$
$$=\frac{1}{9}+\frac{1}{9}\left(\frac{1}{10}\right)+\frac{1}{9}\left(\frac{1}{10}\right)^2+\frac{1}{9}\left(\frac{1}{10}\right)^3+\dots$$
$$=\frac{\frac{1}{9}}{1-\frac{1}{10}}$$
$$=\frac{10}{81}$$