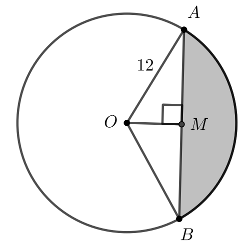
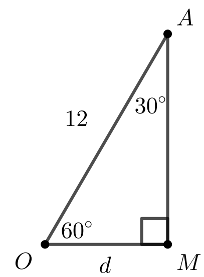
Intermediate Math Circles
Circles and Area
Problem Set 2 Solutions

Exercise 6 Solution

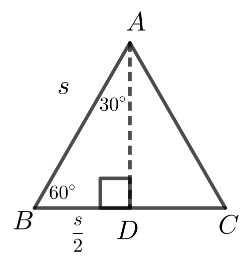
Construct the altitude from O to AB. Label the foot of the altitude M. We now know that OM is perpendicular to AB, $AM = MB$, We also know $\angle AOB = \angle BOM = \frac{120}{2} = 60°$.



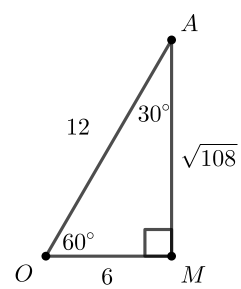
If we take a close look at $\triangle AOM$. We know that $\angle AOM = 60°$ and $\angle AMO = 90°$. This means $\angle OAM = 30°$. We also know that $AM = 12$. Label OM as d.



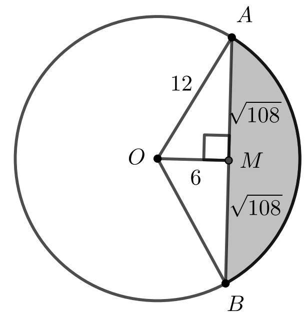
If we look at the process we used to get the formula for the area of an equilateral triangle. We know $\angle ABD = 60°$. We know that when we created the altitude we. bisected $\angle BAC$. That is, $\angle BAD = 30°$. We notice in a triangle with angles $30°, 60°$ and $90°$, the side between the $60°$ and the right angle equals half the length of the hypotenuse.



So in $\triangle AOM$, we know $AO = \frac{12}{2} = 6$. Then using the pythagorean theorem, $AM^2 = 12^2 - 6^2 = 144 - 36 = 108.$. Since $AM > 0$ then $AM = \sqrt{108}$.



We can similarly in $\triangle BOM$, we can show $BO = 6$ and $MB = \sqrt{108}$. We can now find the are of $\triangle AOB$. The base is $2\sqrt{108}$ and the height is 6. Therefore, the area of the triangle is $\frac{(2\sqrt{108})(6)}{2} = 6\sqrt{108}$.



Now the area of the shaded region equals the area of the sector minus the area of the triangle.

$$\text{A} = \frac{120}{360}(12^2\pi) - 6\sqrt{108} = 48\pi - 6\sqrt{108}.$$

NOTE: We can rewrite $\sqrt{108}$ as $\sqrt{36}\sqrt{3} = 6\sqrt{3}$ and the area becomes

$$\text{A} = 48\pi - 6\sqrt{108} = 48\pi - 6(6\sqrt{3}) = 48\pi - 36\sqrt{3}$$

For the past contest questions, the solutions can be found on our webpage. Each url directs you to the contest solutions but you will need find the specific question.

a) 2020 Cayley # 16

https://cemc.uwaterloo.ca/contests/past_contests/2020/2020CayleySolution.pdf

b) Pascal 2013 # 18

https://cemc.uwaterloo.ca/contests/past_contests/2013/2013PascalSolution.pdf

c) 2018 Pascal # 23

https://cemc.uwaterloo.ca/contests/past_contests/2018/2018PascalSolution.pdf