# Intermediate Math Circles
# Counting
# Problem Set Solution

**Exercise 1**: Three-digit positive integers such as 789 and 998 use no digits other than 7, 8, and 9. In total, how many three-digit positive integers use no digits other than 7, 8, and 9?

**Solution 1**:
For each digit there are three options. This means that the number of integers is $3 \times 3 \times 3 = 27$

**Exercise 2**: If you wrote down all of the integers from 3000 to 5999, inclusive, how many times would you write the digit 5?

**Solution 2**:
To answer, you need to count how many times you would write the digit 5 when writing the integers from 3000 to 5999. We will use an organized count.

**First consider the units digit.**
For every cycle of the 10 digits from 0-9, the digit 5 occurs once. There are $3000 \div 10 = 300$ cycles of the digits 0-9. Therefore, there are 300 integers with a 5 in the units digit when writing the integers from 3000 to 5999.

**Next, consider the tens position.**
For every cycle of the 100 digits 00-99, the digit 5 will occur 10 times in the tens position. (These are in the integers 50-59). There are $3000 \div 100 = 30$ cycles of 00-99. Therefore, there are $10 \times 30 = 300$ integers that have the digit 5 in the tens position when writing the integers from 3000 to 5999.

**Next, consider the hundreds position.**
For every cycle of the 1000 digits 000-999, the digit 5 will occur 100 times in the hundreds position. (These are the integers from 500-599). There are $3000 \div 1000 = 3$ cycles of 000-999. Therefore, there are $100 \times 3 = 300$ integers that have the digit 5 in the tens position when writing the integers from 3000 to 5999.

**Finally, consider the thousands position.**
There are 1000 integers that have a 5 in the hundreds position when writing the integers from 3000 to 5999. (These are 5000-5999).

**Now, combine the counts.**
Combining the counts for the four different positions, when writing the integers from 3000 to 5999 the digit 5 will be written a total of $300 + 300 + 300 + 1000 = 1900$ times.

**Exercise 3**: How many of the integers from 3000 to 5999, inclusive, contain at least one 5?

**Solution 3**:
There are two cases we need to look at:

- **Case 1:** The integer contains the digit 5 at least once.
- **Case 2:** The integer doe not contain the digit 5.

We will find the number of integers that do not have a 5 and subtract it from the total number of possible integers.

For the integers that do not have a 5, there are two choices for the first digit. They are 3 or 4. There are nine choices for each of the following three digits. They are 0, 1, 2, 3, 4, 6, 7, 8, or 9.

This means the number of integers from 3000 to 5999, inclusive, in which the digit 5 does not occur is $2 \times 9 \times 9 \times 9 = 1458$.

Therefore, the number of the total number the integers from 300 to 599, inclusive, contain the digit 4 at least once is 3000 - 1458 = 1542.

**Exercise 4**: Consider the integers from 400 to 899, inclusive. How many of these integers have at most one digit that is a 3?

**Solution 4**:

To count the number integers that have at most one digit that is a 3, we need to look at two cases.

- **Case 1:** The digit 3 does not occur in the integer.
- **Case 2:** The digit 3 occurs exactly once in the integer.

**Case 1:** The digit 3 does not occur in the integer.

There are five choices for the first digit. These digits are 4, 5, 6, 7, or 8.
There are nine choices for the other each other digit. These digits are 0, 1, 2, 4, 5, 6, 7, 8, or 9.
This means there are $5 \times 9 \times 9 = 405$ integers that do not have 3 as one of its digits.
The total for **Case 1:** is 405.

**Case 2:** The digit 3 occurs exactly once in the integer.

There are now three cases for this case.

- **Case a:** The first digit is 3 and the other two are not 3.
- **Case b:** The second digit is 3 and the other two are not 3.
- **Case c:** The third digit is 3 and the other two are not 3.

**Case a:** The first digit is 3 and the other two are not 3.

The first digit cannot be 3 since the only choices are 4,5,6,7 or 8.
This means that we have no integers that have the first digit is 3 and the other two are not 3.

**Case b:** The second digit is 3 and the other two are not 3.

There are five choices for the first digit. These digits are 4, 5, 6, 7 or 8. There are nine choices for the third digit. These digits are 0, 1, 2, 4, 5, 6, 7, 8, or 9.
This means there are $5 \times 1 \times 9 = 45$ integers where the second digit is 3 and the other two are not 3.

**Case c:** The third digit is 3 and the other two are not 3.

There are five choices for the first digit. There are nine choices for the second digit.
This means there are $5 \times 9 \times 1 = 45$ integers where the third digit is 3 and the other two are not 3.

The total for **Case 2:** is $0 + 45 + 45 = 90$.

Therefore, there are $405 + 90 = 495$ integers from 400 to 899, inclusive, that have at most one digit that is a 3.

**Exercise 5**: How many three-digit positive integers have exactly one even digit?

**Solution 5**:

There are three cases we need to look at.

- **Case 1:** The first digit is even and the other two are odd.
- **Case 2:** The second digit is even and the other two are odd.
- **Case 3:** The third digit is even and the other two are odd.

**Case 1:** The first digit is even and the other two are odd.
If the first digit is even , there are four choices for this digit. These digits are 2, 4, 6, or 8. (The first digit cannot be 0.) There are five choices for each of the other digits. These digits are 1, 3, 5, 7, or 9. This means that there are $4 \times 5 \times 5 = 100$ integers for **Case 1**.

**Case 2:** The second digit is even and the other two are odd.
If the second digit is even , there are 5 choices for this digit. Since the other digits are odd, there are five choices for each of the other digits. This means that there are $5 \times 5 \times 5 = 125$ integers for **Case 2**.

**Case 3:** The third digit is even and the other two are odd.
If the third digit is even, there are five choices for this digit. Since the other digits are odd, there are five choices for each of the other digits. This means that there are $5 \times 5 \times 5 = 125$ integers for **Case 3**.

Therefore, there are a total of $100 + 125 + 125 = 350$ three-digit positive integers that have exactly one even digit.

**Exercise 6**: If you wrote down all of the integers from 300 to 599, inclusive, what is the sum of all of the digits that you would write?

**Solution 6**:
We give two possible ways to calculate this sum.
*Approach 1:*
First, consider the units digits of the integers from 300 to 399. In each set of ten consecutive integers in this range, each of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ appears once (in the units position) and their sum is 45. There are ten distinct sets of 10 consecutive integers in the range from 300 to 399, and so the sum of the units digits in this range is $10 \times 45 = 450$. Similarly, in both the ranges from 400 to 499 and 500 to 599, the sum of the units digits will be 450. Thus, from 300 to 599, the sum of the units digits will be $3 \times 450 = 1350$.
Now consider the tens digits of the integers from 300 to 399. We note that each of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ appears 10 times in the tens position in this list of 100 integers. For example, the digit 0 appears as the tens digit in the integers from 300 to 309. The sum of these tens digits is $10 \times 45 = 450$. As with the units digits, the same number of each digit occurs again in the ranges 400 to 499 and 500 to 599. Thus, from 300 to 599, the sum of the tens digits will be $3 \times 450 = 1350$.
In the range 300 to 599 there are 100 integers with a 3 in the hundreds position, 100 integers with a 4 in the hundreds position, and 100 integers with a 5 in the hundreds position. The sum of these 300 digits is $100 \times 3 + 100 \times 4 + 100 \times 5 = 1200$.
Therefore, the sum of all digits used in writing down the integers from 300 to 599, inclusive, is

$$1350 + 1350 + 1200 = 3900$$

*Approach 2:*
You may have noticed that you can use your work from Problem 2 in the presentation to calculate the sum in this exercise. We outline how to do this here.
Recall that when writing the integers from 300 to 599, inclusive, the digit 4 will be written a total of 160 times. A similar argument can be used to show that the digits 3 and 5 will also appear a total of 160 times. Convince yourself that the digits 3 and 5 will appear exactly as often as the digit 4 in each of the three positions: hundreds, tens, and units.
The counts for the remaining digits are a bit different because of the particular range of integers. Notice that the digits 1, 2, 6, 7, 8 and 9 do not appear as hundreds digits.
Using the work done in Problem 2 in the presentation, we can see that the digit 9 will appear $30 + 30 = 60$ times. This is because the digit 9 appears exactly as often as the digit 4 in the units position and in the tens position, but does not appear at all in the hundreds position. (Remember that the digit 4 appears 30 times in the tens position and 30 times in the units position.) Similar reasoning shows that the digits 1, 2, 6, 7, and 8 also appear 60 times each.
When we add up all of the digits, each digit 4 contributes 4 to the sum, and so the 160 4s contribute a total of $160 \times 4 = 640$ to the sum of the digits. Dealing with the other digits in a similar way, we can calculate the sum of all of the digits as follows:

$$60 \times 1 + 60 \times 2 + 160 \times 3 + 160 \times 4 + 160 \times 5 + 60 \times 6 + 60 \times 7 + 60 \times 8 + 60 \times 9 = 3900$$

**Exercise 7**: How many positive integers are there between 10 and 1000 having the property that the sum of its digits is 3?

**Solution 7**:

We note that the sum of the digits of 1000 is not 3. Every other positive integer in the given range has two or three digits.

For the sum of the digits of an integer to be 3, no digit can be greater than 3.

If a two-digit integer has sum of digits equal to 3, then its tens digit is 1, 2, or 3. The possible integers are 12, 21, and 30.

If a three-digit integer has sum of digits equal to 3, then its hundreds digit is 1, 2, or 3. If the hundreds digit is 3, then the units and tens digits add to 0, so must be each 0. The integer must thus be 300.

If the hundreds digit is 2, then the units and tens digits add to 1, so must be 1 and 0 or 0 and 1. The possible integers are 210 and 201.

If the hundreds digit is 1, then the units and tens digits add to 2, so must be 2 and 0, or 1 and 1, or 0 and 2, giving possible integers 120, 111, and 102.

Overall, there are 9 such positive integers.

**Exercise 8**: An integer is defined to be *upright* if the sum of its first two digits equals its third digit. For example, 145 is an upright integer since $1 + 4 = 5$. How many positive three-digit integers are upright?

**Solution 8:**

We will do an organized count for this question.

If the last digit is 9 the numbers can be 909, 819, 729, 639, 549, 459, 369, 279, 189. There are 9 integers.

We can set up a table to help count

| Last Digit | First two digits | Number of Integers |
|---|---|---|
| 9 | 90, 81, 72, 63, 54, 45, 36, 27, 18 | 9 |
| 8 | 80, 71, 62, 53, 44, 35, 26, 17 | 8 |
| 7 | 70, 61, 52, 43, 34, 25, 16 | 7 |
| 6 | 60, 51, 42, 33, 24, 15 | 6 |
| 5 | 50, 41, 32, 23, 14 | 5 |
| 4 | 40, 31, 22, 13, | 4 |
| 3 | 30, 21, 12 | 3 |
| 2 | 20, 11 | 2 |
| 1 | 10 | 1 |

Therefore, there are $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$ three-digit upright integers.