Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

October 21, 2020

Graph Theory Proofs

Introduction

Graph theory is a field of mathematics that looks to study objects called graphs. The ideas and understanding gained from studying graphs can be applied to many other problems. Examples of these problems include matching organ donors to patients, finding the best routes from point A to B (like a GPS does) and even some of the problems that we looked at last week in the BCC prep. The key to the relation between problems and graphs is having rules that we know to be true about graphs, called theorems. This week we will look at different aspects of graphs and how we can show that certain things are true about them. In other words, how we can prove traits of certain graphs. First, we need to understand some basic properties of graphs.

Definitions

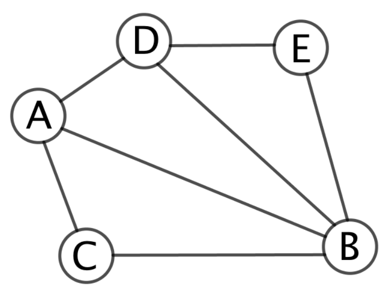


Figure 1: Graph used in examples below.

- A **vertex (plural: vertices)** is a point on a graph. Typically, it is represented by a small circle. It may also have a label.
  Example: The circles labelled A, B, C, D and E in Figure 1 are vertices.

- An **edge** is a line that connects two vertices. The **endpoints** of an edge are the vertices that it connects.
  Example: The segment connecting vertex A and vertex B is an edge. We typically write an edge, e, as $e = \{A, B\}$.

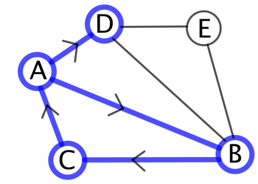
- A **graph** a graph is a collection of vertices with edges between some pairs of them.
  Example: Figure 1 shows a graph.

- A **walk** is a sequence of vertices and edges that lead from one vertex to another. A **path** is a walk with no vertex repeated, except possibly the starting and ending vertex.
  Example: $A, \{A, B\}, B, \{B, C\}, C, \{C, A\}, A, \{A, D\}, D$ is a walk in the example graph.
  $A, \{A, B\}, B, \{B, C\}, C$ is a path.

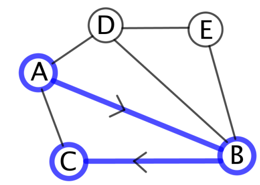
- A **cycle** is a path that has the same starting and ending vertex. The length of a cycle is the number of distinct vertices in the path.
  Example: $A, \{A, B\}, B, \{B, C\}, C, \{C, A\}, A$ represents a cycle of length 3 in the graph to the right.

- The **neighbours** of a vertex are all vertices that are connected to the vertex by an edge.
  Example: The neighbours of vertex C are A and B.

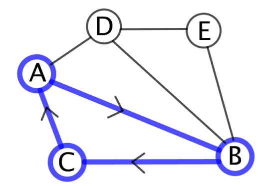
- The **degree** of a vertex is the number of neighbours that it has.
  Example: Vertex C has degree 2.



Walk



Path



Cycle

This week we will be looking only at **simple graphs**. A **simple graph** is a graph with three specific conditions.

1. Edges have no direction (we can travel in both directions on them when creating a walk or path).

2. There is most one edge between two vertices.

3. There are no loops. In other words, no edges that connect back to the same vertex.

Exercise 1. Throughout this document there are online exercises to help practice what we have learned. Click on the link to try the activity: *https://www.geogebra.org/m/qeajsqya*

In order to talk about graph theory proofs, we also need to understand what a proof is.

Definition 1. *Proof*
A mathematical proof shows conclusive evidence that a mathematical statement is true. Once a statement has an accepted proof, it can be seen as a fact.

Planarity

Definition 2. *Planar embedding*
A planar embedding is a way to draw a graph where no edges and no vertices overlap.

Definition 3. *Planar Graph and Non-Planar Graph*
A planar graph is a graph that has a planar embedding. A non-planar graph has no planar embedding.

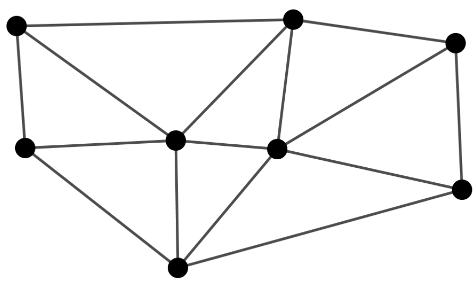


Figure 2: A planar embedding of a graph.

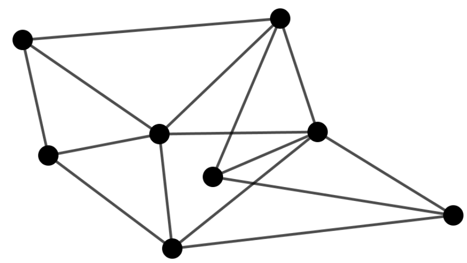


Figure 3: A non-planar embedding of the graph in Figure 2. Note that even if a graph is planar, not all of the ways that we can draw it will be a planar embedding.

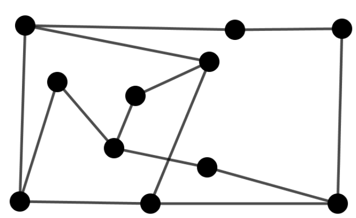


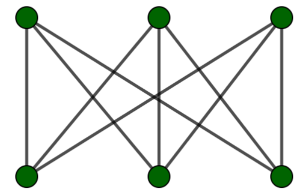
Figure 4: A non-planar graph. There is no planar embedding of this graph. No matter how we arange the vertices, we will always have edges that overlap. *Note: We will see this graph again in the Problem Set, where you can prove that it is indeed non-planar.*

Exercise 2. Find a planar embedding: *https://www.geogebra.org/m/rtyvz69a*

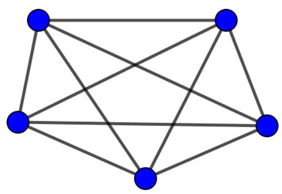
We can prove that a graph has a planar embedding (and so it is planar) by drawing a picture of the planar embedding. If we can show the planar embedding, then it must exist. But how do we prove that a graph is non-planar? There a few theorems mathematicians use, but one definitive theorem is Kuratowski's Theorem. To understand Kuratowski's Theorem, we need a few more definitions.

Definition 4. *$K_{3,3}$*

The $K_{3,3}$ graph is a graph with 6 vertices, divided into 2 groups. Every vertex from each group is connected to every vertex in the other group. In the image given, the top three vertices are one group and the bottom three are another.



$K_{3,3}$



K_5

Definition 5. *K_5*

The K_5 graph is a graph with 5 vertices, where each vertex connects with each other vertex.

The $K_{3,3}$ and K_5 graphs are both non-planar. This online activity: *https://www.geogebra.org/m/pktcgcz6* will allow you to move them around to convince yourself of this. Note that this is not a proof, the proof that these are non-planar requires higher-level mathematics that you will see in university. Rather than prove this fact, we're going to use it to prove other graphs are non-planar.

Definition 6. *Edge Subdivision*

We can create an edge subdivision of a graph by replacing each of its edges by a path. Watch this video: *https://youtu.be/iVFrR2ks9TA* to learn more. We have now learned what we need to understand Kuratowski's Theorem.

Theorem 1. *Kuratowski's Theorem*

A graph is planar if does not have an edge subdivision of either the $K_{3,3}$ or the K_5 graph as a part of the graph. If it does have such an edge subdivision, it is non-planar.

In other words, we can show that a graph is non-planar by finding either a $K_{3,3}$ or K_5 edge subdivision within the graph.

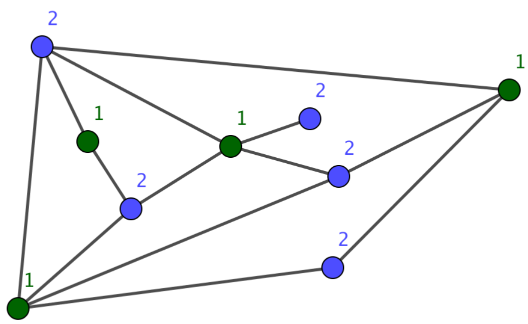
Watch this video: *https://youtu.be/OCegRy-xhjQ* for examples of using Kuratowski's Theorem to prove a graph is non-planar.

Then, use the following online activities to practice finding a $K_{3,3}$: *https://www.geogebra.org/m/khnvzzmn* and a K_5: *https://www.geogebra.org/m/ufzpbfvy* edge subdivision, thus proving by Kuratowski's Theorem that the graphs are non-planar.

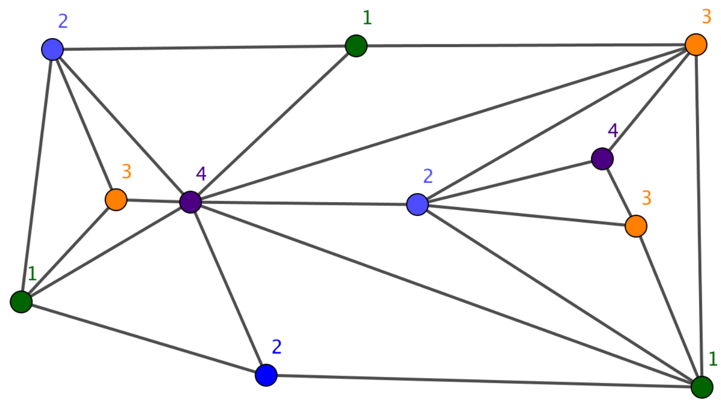
Colouring

Definition 7. *k-Colouring*
A k-colouring of a graph is an assignment of at most k different colours so that each vertex is assigned a colour. Each vertex must have a different colour than any of its neighbours.

Examples of colourings:



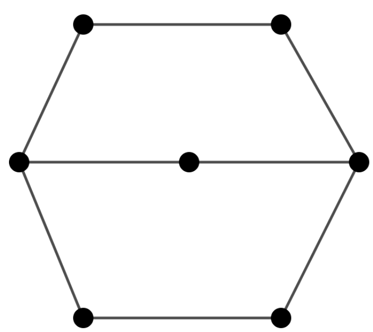
2–Colouring



4–Colouring

Exercise 3. Find a 3-colouring: *https://www.geogebra.org/m/eddgdxh5*

An interesting question is the least number of colours that we can colour a graph with. This is called a graph's **chromatic number**. We can prove that something is k-colourable by showing an example of a colouring. But how do we show that something is not k-colourable? Consider the following graph:



It is not 2-colourable.

Use this online activity: *https://www.geogebra.org/m/vu3rgygj* to convince yourself of this.

Think about what makes this graph not 2-colourable. Watch the video: *https://youtu.be/r6VhIWYFVyw* for a full explanation.