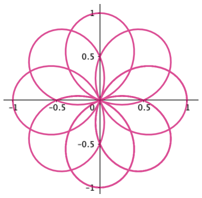
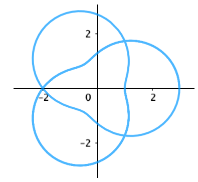


CEMC Math Circles - Grade 11/12
October 14 - 20, 2020
Polar Coordinates - Solution



Warm-up Questions:

(a) Convert the angles with the following measures from degrees to radians: $180°$, $90°$, $60°$, $45°$, $30°$, $48°$.

Answers: $180° = \pi$, $90° = \dfrac{\pi}{2}$, $60° = \frac{\pi}{3}$, $45° = \dfrac{\pi}{4}$, $30° = \dfrac{\pi}{6}$, $48° = \dfrac{4\pi}{15}$.

(b) Convert the angles with the following measures from radians to degrees: $\dfrac{\pi}{5}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}, \dfrac{7\pi}{4}$.

Answers: $\dfrac{\pi}{5} = 36°$, $\dfrac{5\pi}{6} = 150°$, $\dfrac{3\pi}{2} = 270°$, $\dfrac{7\pi}{4} = 315°$.

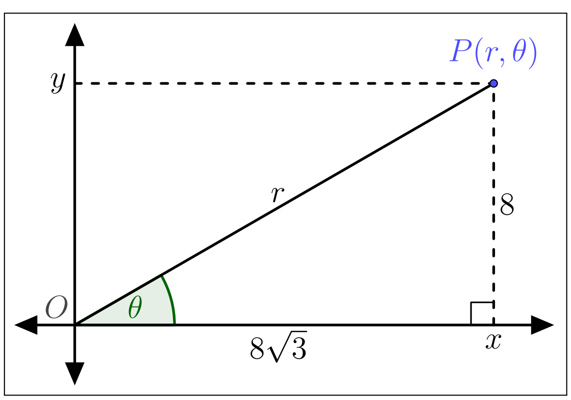
(c) Complete the chart below. *The angles are given in radians.*

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Question 1

Plot the points with Cartesian coordinates $A\big(8\sqrt{3}, 8\big)$ and $B\big(\frac{5}{4}, \frac{5\sqrt{3}}{4}\big)$ and then convert them to polar coordinates.

Solution: We first plot the point $A\big(8\sqrt{3}, 8\big)$ in the plane.



Since $x = 8\sqrt{3}$ and $y = 8$, we have

$$r = \sqrt{x^2 + y^2} = \sqrt{\big(8\sqrt{3}\big)^2 + 8^2} = \sqrt{192 + 64} = 16$$

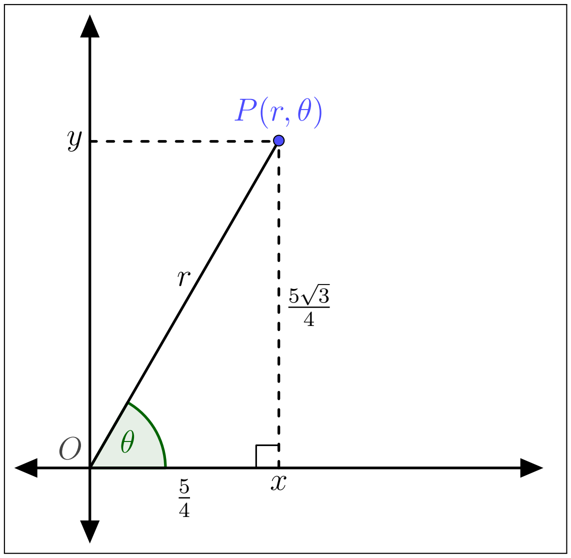
From the right-angled triangle in the diagram, we see we are looking for an angle θ in the first quadrant that satisfies

$$\sin\theta = \frac{y}{r} = \frac{8}{16} = \frac{1}{2}$$

One possible choice is $\theta = \frac{\pi}{6}$.

This means the point with Cartesian coordinates $(x, y) = \big(8\sqrt{3}, 8\big)$ can be described using polar coordinates $(r, \theta) = \big(16, \frac{\pi}{6}\big)$.

Now we plot the point $B\big(\frac{5}{4}, \frac{5\sqrt{3}}{4}\big)$ in the plane.



Since $x = \frac{5}{4}$ and $y = \frac{5\sqrt{3}}{4}$, we have

$$x^2 + y^2 = \left(\tfrac{5}{4}\right)^2 + \left(\tfrac{5\sqrt{3}}{4}\right)^2 = \tfrac{25}{16} + \tfrac{75}{16} = \tfrac{100}{16} = \tfrac{25}{4}$$

and so $r = \sqrt{x^2+y^2} = \frac{5}{2}$. From the right-angled triangle in the diagram, we see we are looking for an angle θ in the first quadrant that satisfies

$$\cos\theta = \frac{x}{r} = \frac{\left(\frac{5}{4}\right)}{\left(\frac{5}{2}\right)} = \frac{1}{2}$$

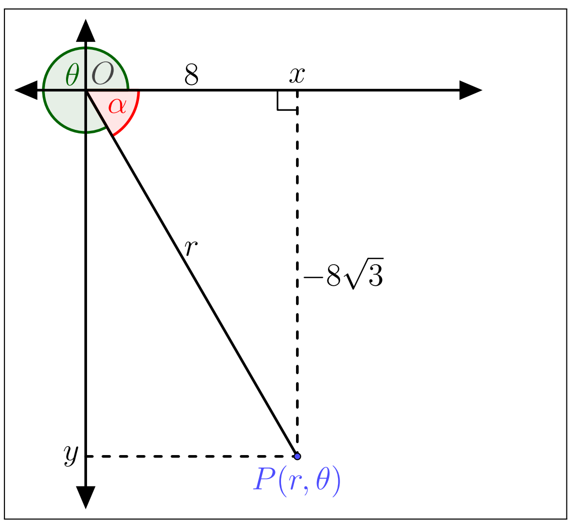
One possible choice is $\theta = \frac{\pi}{3}$.

This means the point with Cartesian coordinates $(x, y) = \left(\frac{5}{4}, \frac{5\sqrt{3}}{4}\right)$ can be described using polar coordinates $(r,\theta) = \left(\frac{5}{2}, \frac{\pi}{3}\right)$.

Question 2

Plot the points with Cartesian coordinates $C\big(8, -8\sqrt{3}\big)$ and $D\big(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}\big)$ and then convert them to polar coordinates.

Solution: We first plot the point $C\big(8, -8\sqrt{3}\big)$ in the plane.



Since $x = 8$ and $y = -8\sqrt{3}$, we have

$$r = \sqrt{x^2+y^2} = \sqrt{8^2 + \big(-8\sqrt{3}\big)^2} = \sqrt{64+192} = 16$$

From the right-angled triangle in the diagram, we see we are looking for an angle θ in the fourth quadrant for which the associated acute angle α satisfies

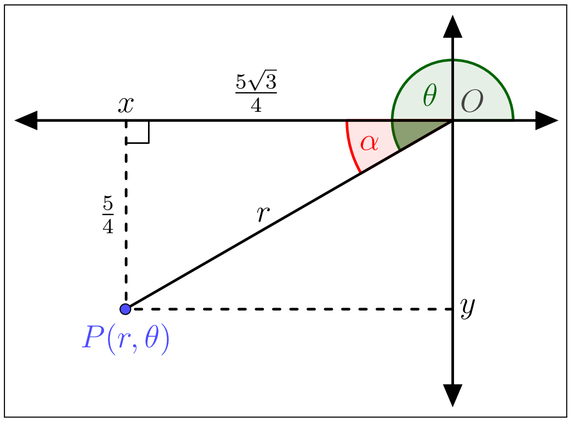
$$\cos\alpha = \frac{8}{16} = \frac{1}{2}$$

This means $\alpha = \frac{\pi}{3}$ and so one possible choice is $\theta = \frac{5\pi}{3}$.

This means the point with Cartesian coordinates $(x, y) = \big(8, -8\sqrt{3}\big)$ can be described using polar coordinates $(r,\theta) = \big(16, \frac{5\pi}{3}\big)$.

Note: We could have instead observed that point C is related to point A. They are the same distance from the origin, and their angles are complementary.

Now we plot the point $D\big(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}\big)$ in the plane.



Since $x = -\frac{5\sqrt{3}}{4}$ and $y = -\frac{5}{4}$, we have

$$x^2 + y^2 = \left(-\frac{5\sqrt{3}}{4}\right)^2 + \left(-\frac{5}{4}\right)^2 = \frac{75}{16} + \frac{25}{16} = \frac{25}{4}$$

and so $r = \sqrt{x^2+y^2} = \frac{5}{2}$. From the right-angled triangle in the diagram, we see we are looking for an angle θ in the third quadrant for which the associated acute angle α satisfies

$$\sin\alpha = \frac{\left(\frac{5}{4}\right)}{\left(\frac{5}{2}\right)} = \frac{1}{2}$$

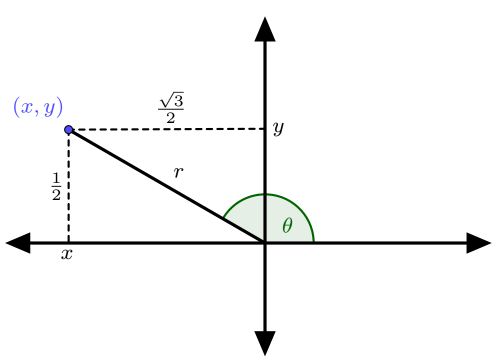
This means $\alpha = \frac{\pi}{6}$ and one possible choice is $\theta = \frac{7\pi}{6}$.

This means the point with Cartesian coordinates $(x,y) = \left(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$ can be described using polar coordinates $(r,\theta) = \left(\frac{5}{2}, \frac{7\pi}{6}\right)$.

Question 3

Plot the point with Polar coordinates $P(-1, \frac{11\pi}{6})$ and then convert it to Cartesian coordinates.

Solution: First we represent $P(-1, \frac{11\pi}{6})$ with a positive value of r. In other words, it is the point $P(r,\theta) = P(1, \frac{5\pi}{6})$. This is because this point lies on the line passing through the origin and making an angle of $\frac{5\pi}{6}$. Also, the negative sign of r means that we move in the direction *opposite* to the direction defined by $\theta = \frac{11\pi}{6}$.



Using the expressions of x and y in terms of r and θ, we see that

$$x = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$
$$y = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

Therefore, the point has Cartesian coordinates $(x,y) = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$.

Activity Answers:

In a few weeks we will learn how to...

G	R	A	P	H	P	O	L	A	R	C	U	R	V	E	S!
12	8	7	11	9	11	4	2	7	8	6	3	8	10	1	5

We now provide an explanation of each matching.

1. This point has polar coordinates $(4,0)$. ***(E)***

   Since $r = 4$ and $\theta = 0$, this is a point that is 4 units from the origin and lies on the ray defined by $\theta = 0$ which is the positive x-axis. This describes only point E.

2. This point has polar coordinates $\left(4, \frac{3\pi}{2}\right)$. ***(L)***

This is a point that is 4 units from the origin and lies on the ray defined by $\theta = \frac{3\pi}{2}$ *which is the negative* y*-axis. This describes only point* L*.*

3. This point has polar coordinates $\left(4, \frac{3\pi}{4}\right)$. ***(U)***

This is a point that is 4 units from the origin lies on the ray defined by $\theta = \frac{3\pi}{4}$*. This describes only point* U*.*

4. This point could also be described using polar coordinates $\left(2, \frac{11\pi}{4}\right)$. ***(O)***

Note that $\frac{11\pi}{4}$ *and* $\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}$ *are equivalent angles. So we are looking for the point with polar coordinates* $\left(2, \frac{3\pi}{4}\right)$*. This is on the same ray as* U *above, but 2 units from the origin. This describes only point* O*.*

5. This point's first coordinate, r, satisfies $r^2 = 2$. ***(S)***

This means $r = \pm\sqrt{2} \approx \pm 1.4$*. It looks like the only point that is around 1.4 units from the origin is* S*. You can draw a circle of radius* 1.4 *on the graph to confirm. This is describing point* S*.*

6. This point has the largest first coordinate, r, out of all of the points. ***(C)***

The point with the largest first coordinate will be the farthest from the origin. The point C *is 5 units away and every other point appears to be closer than that. You can draw a circle of radius* 5 *on the graph to confirm! This is describing point* C*.*

7. This point has the smallest *positive* second coordinate, θ, out of all of the points. ***(A)***

The point with the smallest positive second coordinate will make the smallest angle with the positive x*-axis. This describes the point* A*.*

8. This point's second coordinate, θ, satisfies $2\sin\theta = 1$. ***(R)***

If $0 \le \theta < 2\pi$ *and* $\sin\theta = \frac{1}{2}$*, then* $\theta = \frac{\pi}{6}$ *or* $\theta = \frac{5\pi}{6}$*. The only point that lies on the ray defined by* $\theta = \frac{\pi}{6}$ *is* R *and there are no points that lie on the ray defined by* $\theta = \frac{5\pi}{6}$*. This describes* R*.*

9. This point's second coordinate, θ, satisfies $\cos\theta = -1$. ***(H)***

If $0 \le \theta < 2\pi$ *and* $\cos\theta = -1$*, then* $\theta = \pi$*. The only point that lies on the ray defined by* $\theta = \pi$ *(the negative* x*-axis) is* H*.*

10. This point's first coordinate, r, satisfies $r = 3$. ***(V)***

The only point that appears to be 3 units from the origin is V*. You can draw a circle of radius* 3 *on the graph to confirm. This is describing point* V*.*

11. This point's coordinates satisfy $r = \sin\theta$. ***(P)***

Since $-1 \le \sin\theta \le 1$*, any coordinates that satisfy this equality must have* $-1 \le r \le 1$*. The only point within 1 unit of the origin is* P*. In fact,* P *appears to have polar coordinates* $r = 1$ *and* $\theta = \frac{\pi}{2}$ *which do satisfy* $\sin\theta = \sin\left(\frac{\pi}{2}\right) = 1 = r$*.*

12. This point's coordinates satisfy $r = \theta$. ***(G)***

We are now left with one property (12) and one point (G*). This means* G *must be the point satisfying* $r = \theta$*. Using the distance formula, you can check that* G *is around 4 units from the origin. The ray through* G *is near the ray defined by* $\theta = \frac{5\pi}{4} \approx 4$*, which provides some evidence that* $r \approx \theta$*. (The actual point plotted has* $r = \theta = 4.1$*.)*