Intermediate Math Circles
October 27 2021
Problem Set 1

1. Let $a = \angle EGF$, $b = \angle FEG$. Since $AB \parallel CD$, we know that alternate interior angles are equal so $a = 50°$. Observe that a and the angle $13x$ form a straight angle. Then,

$$\begin{aligned} a + 13x &= 180 \\ 50 + 13x &= 180 \\ x &= 10 \end{aligned}$$

Similarly, using b, the $50°$ angle, and the angle $3x = 30$,

$$\begin{aligned} b + 50 + 30 &= 180 \\ b + 80 &= 180 \\ b &= 100 \end{aligned}$$

Since y is an external angle to $\triangle EFG$, $y = a + b = 150°$.
Therefore, $x = 10°, y = 150°$.

2. $\triangle ABD$ is isosceles since $AB = BD$. Therefore $\angle BDA = \angle BAD = 52°$.

Then in $\triangle BAD$,

$$\begin{aligned} \angle ABD &= 180° - \angle A - \angle BDA \\ &= 180° - 52° - 52° \\ &= 76° \end{aligned}$$

Since AB $\parallel$ DC, we have $\angle BDC = \angle ABD = 76°$.

Since BD = BC, $\triangle BDC$ is isosceles. Therefore, $\angle BDC = \angle BCD = 76°$

Therefore, by sum of interior angles of a triangle, $\angle DBC = 180° - 76° - 76° = 28°$.

3. In a square, the corner angles are $90°$. The triangle is equilateral (all sides equal), so we know all the angles are equal and hence must be $60°$ each.

If we look at the place where the triangle and two squares meet (where x is located), we notice it is made up of four angles; two corner angles of a square, one corner angle of a triangle, and x. These four angles form a complete revolution, so they must sum up to $360°$.

Then,

$$\begin{aligned} x + 90 + 90 + 60 &= 360 \\ x + 240 &= 360 \\ x &= 120° \end{aligned}$$

Therefore the measure of angle x is $120°$.

4. Since $\angle FJI = 111°$ is part of a straight angle with $\angle FJG$, we have that $\angle FJG = 69°$.

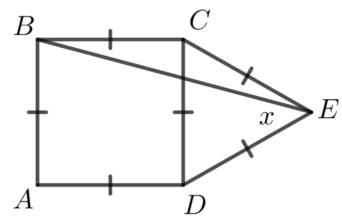
   We see that because $GF = GJ$, $\triangle FGJ$ is isosceles, with equal base angles $\angle FJG$ and $\angle GFJ$, we get $\angle GFJ = 69°$ and so $\angle FGJ = 42°$

   Because $FG \parallel IH$, $\angle FGI = \angle GIH = 42°$. Also, $\triangle IHG$ is isosceles since $GH = HI$, so $\angle IGH = \angle GIH = 42°$

   Since $GH \parallel FI$, $\angle FIG = \angle IGH = 42°$.

   Using $\triangle FJI$, we see
   $$\begin{aligned} \angle FJI + \angle FIJ + \angle JFI &= 180 \\ 111 + 42 + \angle JFI &= 180 \\ \therefore \angle JFI &= 27° \end{aligned}$$

5. Since $ABCD$ is a square, $BC = CD$. Since $\triangle CDE$ is equilateral, $CD = DE = EC$. Therefore, $BC = CD = DE = EC$ and so $BC = EC$.

   By the properties of a square, $\angle BCD = 90°$. By the properties of equilateral triangles, $\angle DCE = 60°$. Therefore $\angle BCE = \angle BCD + \angle DCE = 90 + 60 = 150°$.

   Since $BC = EC$, $\triangle BCE$ is isosceles. So $\angle EBC = \angle BEC = x$. In this triangle, we have
   $$\begin{aligned} \angle BCE + x + x &= 180 \\ 150 + 2x &= 180 \\ x &= 15° \end{aligned}$$

   So $\angle BEC = x = 15°$.

   Note that $60° = \angle DEC = \angle BED + \angle BEC = \angle BED + 15$.

   Therefore, $\angle BED = 60 - 15 = 45°$.

6. Consider the angles opposite to the angles marked y. Since they are opposite angles, they are equal to y.

   The quadrilateral formed in the overlap must have angle sum $360°$. We know two of the angles are y.

   The other two angles are actually the missing angle of the two isosceles triangles. In the left triangle, this angle is $180 - 2x$; for the triangle on the right, it is also $180 - 2x$.

   These four angles have to sum to $360°$. Therefore,
   $$\begin{aligned} y + y + (180 - 2x) + (180 - 2x) &= 360 \\ 2y + 360 - 4x &= 360 \\ 2y &= 4x \\ y &= 2x \end{aligned}$$

   $\therefore y = 2x$ is our desired relationship.

7. Let $\angle SPR = x$. Then, $\angle QPR = \angle QPS + \angle SPR = 12^\circ + x$.

   Since $PS = SR$, $\triangle SPR$ is isosceles and so $\angle PRS = \angle SPR = x$. Since $PS = PQ$, $\triangle PQS$ is isosceles and so $\angle PQS = \angle PSQ = y$.

   Then

$$\begin{aligned} 12 + y + y &= 180 \\ 2y &= 168 \\ y &= 84^\circ \end{aligned}$$

   Since QSR is a straight line, $y = \angle PSQ$ is external to $\triangle PSR$, so $84^\circ = y = x + x = 2x$.

   Therefore, $x = 42^\circ$. And $\angle QPR = \angle QPS + \angle SPR = 12 + x = 12 + 42 = 54^\circ$.

8. Since we are using angle bisectors, let $\angle LNO = \angle ONM = x$, $\angle NLO = \angle OLM = y$, and $\angle LMO = \angle OMN = z$.

   But $68^\circ = \angle LNM = \angle NLO + \angle OLM = 2x$, so $x = 34^\circ$.

   We also have $\angle LON = 180 - (x + y) = 146 - y$, $\angle LOM = 180 - (y + z)$, and $\angle NOM = 180 - (x + z) = 146 - z$.

   $\angle LON$, $\angle NOM$, and $\angle LOM$ form a complete revolution.

   So, $\angle LOM = 360 - \angle LON - \angle NOM = 360 - (146 - y) - (146 - z) = 68 + y + z$

   Using the entire triangle,

$$\begin{aligned} \angle LNM + \angle NLM + \angle LMN &= 180 \\ 68 + 2y + 2z &= 180 \\ 2y + 2z &= 112 \\ y + z &= 56 \end{aligned}$$

   Therefore, substituting back in, we get $\angle LOM = 68 + 56 = 124^\circ$.

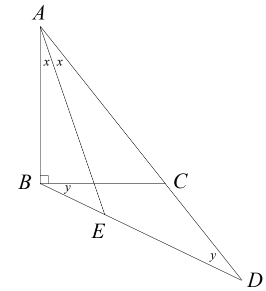
9. Since $AB = AF$, $\triangle ABF$ is isosceles, so $\angle AFB = \angle ABF = a$.
   Since $\angle AFB$ and $\angle DFE$ are opposite angles, $\angle DFE = \angle AFB = a$.

   $\angle ABE$ is external to $\triangle CBE$, so $\angle ABE = \angle ACE + \angle BEC$ and $a = x + z$ follows. (1)

   $\angle ADC$ is external to $\triangle DFE$, so $\angle ADC = \angle DFE + \angle DEC$ and $y = a + z$ follows. (2)

   Substituting (1) into (2) for a, we obtain $y = x + z + z$. Rearranging and simplifying we obtain $x - y + 2z = 0$. This is the equation relating $x,\ y,\ z$.

10. Since AE bisects $\angle BAC$, we can let $x = \angle BAE = \angle EAC$.
Since CB = CD, $\triangle BCD$ is isosceles so $y = \angle CBD = \angle CDB$.



In $\triangle ABD$, by the sum of interior angles of a triangle,

$$\begin{aligned} 2x + (90 + y) + y &= 180 \\ 90 + 2x + 2y &= 180 \\ 2x + 2y &= 90 \\ x + y &= 45 \end{aligned}$$

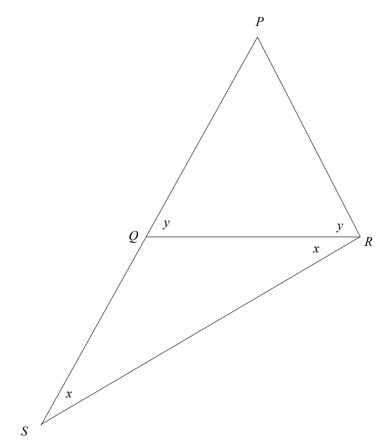
In $\triangle ABE$, using the sum of interior angles,

$$\begin{aligned} x + (90 + y) + \angle AEB &= 180 \\ 90 + (x + y) + \angle AEB &= 180 \\ 90 + 45 + \angle AEB &= 180 && \text{(from above)} \\ 45 + \angle AEB &= 90 \\ \angle AEB &= 45^\circ && \text{(as required)} \end{aligned}$$

11. Since $\triangle PQR$ is isosceles, then $\angle PRQ = \angle PQR = 2x^\circ$.
Since $\angle PRQ$ and $\angle SRT$ are opposite angles, then $\angle SRT = \angle PRQ = 2x^\circ$.
Since $\triangle RST$ is isosceles with $RS = RT$, then

$$\begin{aligned} \angle RST &= \frac{1}{2}(180^\circ - \angle SRT) \\ &= \frac{1}{2}(180^\circ - 2x^\circ) \\ &= (90 - x)^\circ \end{aligned}$$

12. Since $PQ = PR$ and $QS = QR$, we can label the diagram as shown.



Note that $\angle SPR = 180 - 2y$. Using $\triangle SPR$, we see the angle sum gives us

$$\begin{aligned} 180 &= \angle SPR + \angle PSR + \angle PRS \\ 180 &= (180 - 2y) + x + (x + y) \\ 180 &= 180 - y + 2x \\ y &= 2x \end{aligned}$$

So $\angle PRS = x + y = x + 2x = 3x = 3(\angle QSR)$ as required.

13. Since $\angle TQP$ and $\angle RQU$ are opposite angles, then $\angle RQU = \angle TQP = x^\circ$.
Similarly, $\angle QRU = \angle VRS = y^\circ$.
Since the angles in a triangle add to 180°, then
$\angle QUR = 180^\circ - \angle RQU - \angle QRU = 180^\circ - x^\circ - y^\circ$.
Now, $\angle WQP$ and $\angle WQR$ are supplementary, as they lie along a line.
Thus, $\angle WQR = 180^\circ - \angle WQP = 180^\circ - 2x^\circ$.
Similarly, $\angle WRQ = 180^\circ - \angle WRS = 180^\circ - 2y^\circ$.
Since the angles in $\triangle WQR$ add to 180°, then

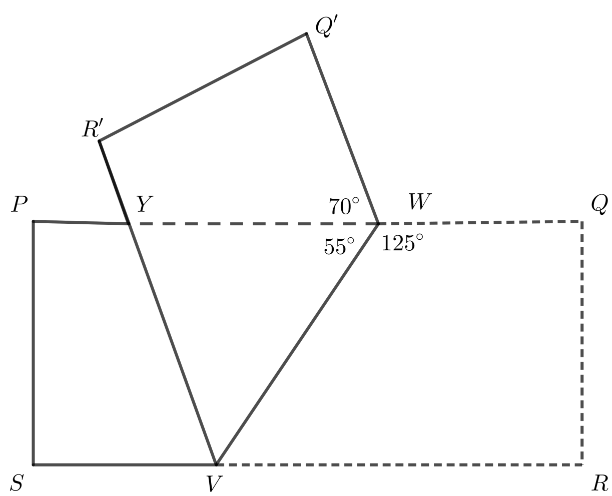
$$\begin{aligned} 38^\circ + (180^\circ - x^\circ) + (180^\circ - 2y^\circ) &= 180 \\ 218^\circ &= 2x^\circ + 2y^\circ \\ x^\circ + y^\circ &= 109^\circ \end{aligned}$$

Finally, $\angle QUR = 180^\circ - x^\circ - y^\circ = 180^\circ - (x^\circ + y^\circ) = 180^\circ - 109^\circ = 71^\circ$.

14. Since points Y, W and Q form a straight line segment, then $\angle YWV = 180^\circ - \angle VWQ$ and so $\angle YWV = 180^\circ - 125^\circ = 55^\circ$.

Since Q' is the final position of Q after folding, then $\angle Q'WV = \angle QWV$.

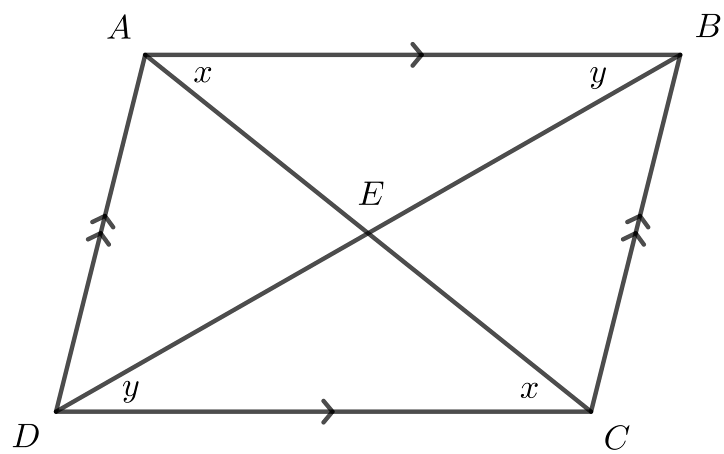
Thus, $\angle Q'WV = \angle QWV = 125^\circ$ and so $\angle Q'WY = \angle Q'WV - \angle YWN = 125^\circ - 55^\circ = 70^\circ$.



Since $Q'W$ and $R'Y$ are parallel sides of the piece of paper, then $\angle R'YW + \angle Q'WY = 180^\circ$, and so $\angle R'YW = 180^\circ - \angle Q'WY = 180^\circ - 70^\circ = 110^\circ$.
Finally, $\angle PYV$ is opposite $\angle R'YW$ so $\angle PYV = \angle R'YW = 110^\circ$.

15. If the diagonals bisect each other the we need to show $AE = EC$ and $BE = ED$.

Here is the parallelogram with certain angles marked.



In $\triangle AEB$ and $\triangle CED$

$$\begin{aligned}\angle EAB &= \angle ECD \text{ (alternate angles)}\\ AB &= CD \text{ (property of a parallelogram)}\\ \angle EBA &= \angle EDC \text{ (alternate angles)}\end{aligned}$$

Therefore, $\triangle AEB \cong \triangle CED$ (ASA congruency)

Therefore:

$AE = EC$ (corresponding sides of congruent triangles)
$BE = ED$ (corresponding sides of congruent triangles)

Therefore the diagonals bisect each other.