Intermediate Math Circles
Wednesday November 3, 2021
Solutions to Problem Set 2

1.) Chord BC subtends both $\angle BAC$ and $\angle BDC$. Therefore, from Circle Theorem 2, $\angle BAC = \angle BDC = 60°$. Now in $\triangle BAE$, $\angle BAE + \angle AEB + \angle ABE = 180$ or $60 + 80 + x = 180$ and $x = 40°$.
Therefore, $\angle ABE = 40°$.

2.) $\triangle AOB$ is isosceles. Therefore, $\angle BAO = 65°$ and $\angle DAB = 35° + 65° = 100°$.
Now $BADC$ is a cyclic quadrilateral, Therefore, $\angle DAB + \angle BCD = 180$ or $100 + \angle BCD = 180$.
Therefore $\angle BCD = 80°$.

3.) Chord BC subtends both $\angle BAC$ and $\angle BDC$. Therefore, from Circle Theorem 2, $\angle BDC = \angle BAC = 45°$. Now in $\triangle FDC$, $\angle FCD + \angle CDF + \angle DFC = 180$ or $\angle FCD + 45 + 95 = 180$ and $\angle FCD = 40°$.
Now quadrilateral $AEDC$ is a cyclic quadrilateral. Therefore, $\angle ACD + \angle DEA = 180$ or $40 + \angle DEA = 180$ Therefore, $\angle DEA = 140°$.

4.) Chord BC subtends both $\angle BAC$ and $\angle BDC$. Therefore, from Circle Angle 2, $\angle BDC = \angle BAC = 40°$. Now $\triangle DOC$ is isosceles.
Therefore $\angle OCD = \angle ODC = 40°$

5.) Since $OB \parallel BC$, then $\angle OBC + \angle DCB = 180$. It follows that $\angle DCB = 115°$. Now $ABCD$ is a cyclic quadrilateral. Therefore, $\angle BAD + \angle DCB = 180$ or $\angle BAD + 115 = 180$.
Therefore, $\angle BAD = 65°$.

6.) Now $\triangle DOC$ is isosceles. Therefore, $\angle ODC + \angle OCD = \frac{180-110}{2} = 35$. Now $ABCD$ is a cyclic quadrilateral. Therefore, $\angle BAD + \angle DCB = 180$ or $80 + \angle BCD = 180$ or $\angle BCD = 100$.
Now $\angle BCD = \angle OCB + \angle OCD$ or $100 = \angle OCB + 35$.
Therefore, $\angle OCB = 65°$.

7.) Since $\angle BDC$ is opposite $\angle FDG$, then $\angle BDC = \angle FDG = 40$. Now $\angle ABD$ and $\angle CDB$ are alternate angles for $BA \parallel CD$. Therefore, $\angle ABD = \angle CDB = 40$. Now, chord AD subtends both $\angle ABD$ and $\angle ACD$.
Therefore, $\angle ACD = \angle ABD = 40°$.

8.) Now $\angle ACB$ is an exterior angle to $\triangle EAC$. Therefore, $\angle ACB = \angle ECA + \angle EAC = 48$. Now, chord CD subtends both $\angle DBC$ and $\angle DAC$. Therefore, $\angle DBC = \angle DAC = 15$. Now $\angle AXB$ is an exterior angle to $\triangle BCX$.
Therefore, $\angle AXB = \angle XBC + \angle XCB = 63$.

9.) Now chord AC subtends the central angle $\angle AOC$ and inscribed angle $\angle ADC$. Therefore by Circle Theorem 1, $\angle ADC = \frac{1}{2}\angle AOC = (2x+5)$. Now $ABCD$ is a cyclic quadrilateral. Therefore,

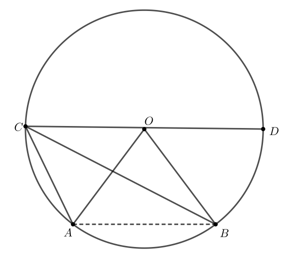
$$\begin{aligned} \angle ABC + \angle ADC &= 180 \\ 3x - 10 + 2x + 5 &= 180 \\ 5x - 5 &= 180 \\ 5x &= 185 \\ x &= 37 \end{aligned}$$

Therefore $\angle ADC = 2(37) + 5 = 79°$.

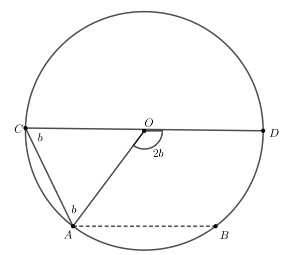
10.) In the circle centred on P, chord ED subtends both $\angle EFD$ and $\angle ECD$. Therefore, from Circle Theorem 2, $\angle EFD = \angle ECD = 40°$. Now $\angle ACB$ is opposite $\angle ECD$. Therefore, $\angle ACB = \angle ECD = 40°$.. Now, In the circle centred on O, chord AB subtends both $\angle AFB$ and $\angle ACB$.
Therefore, from Circle Theorem 2, $\angle AFB = \angle ACB = 40°$.

11.) We will need to show that $\angle AOB = 2\angle ACB$.

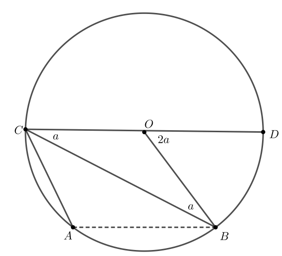
We will construct the diameter from C.



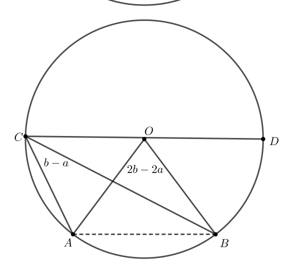
We now look at $\triangle COA$. We know $\triangle COA$ is isosceles. Therefore let $\angle OAC = \angle OCA = b$. Now $\angle AOD$ is exterior to $\triangle COA$ and therefore $\angle AOD = 2b$.



We now look at $\triangle COB$. We know $\triangle COB$ is isosceles. Therefore let $\angle OCB = \angle OBC = a$. Now $\angle BOD$ is exterior to $\triangle COB$ and therefore $\angle BOD = 2a$.

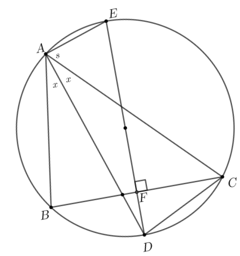


For inscribed ACB, $\angle ACB = \angle OCA - \angle OCB = b - a$. For central angle AOB, $\angle AOB = \angle AOD - \angle BOD = 2b - 2a$.



Now, $\angle AOB = 2b - 2a = 2(b - a) = 2\angle ACB$. Therefore, the central angle subtended by a chord is twice the angle of an inscribed angle subtended by the same chord when the centre of the circle is outside the inscribed angle.

12.) Construct AE and DC. Let the intersection of BC and DE be F and $\angle EAD = s$.
To show that ED is a diameter we will show $\angle DAE = 90°$. (i.e. $x + s = 90°$)



Chord BD subtends both $\angle BAD$ and $\angle BCD$. Therefore, from Circle Theorem 2, $\angle BAD = \angle BCD = x$.

Chord CE subtends both $\angle EAC$ and $\angle EDC$. Therefore, from Circle Theorem 2, $\angle EAC = \angle EDC = s$.

Now in $\triangle FCD, \angle FCD + \angle CDF + \angle DFC = x + s + 90 = 180$ or $x + s = 90$.

Therefore, $\angle DAE = 90°$ and DE is a diameter.