OEIS Math Circles Part 1

In this two part series, we're going to look at a few of my favourite sequences in great depth. Our goals of this learning activity are the following:

(i) An exploration into prime numbers and some interesting sequences relating to prime numbers

(ii) An exploration into recursive sequences, namely the Fibonacci Numbers, Lucas Numbers and others

(iii) Explore `www.oeis.org` and find some new sequences!

Relating to the last point, we will frequently make references to `www.oeis.org` and periodically we will use the numbering scheme used there to refer to sequences. As an example, go to the above website and type in **A000001** for the first sequence in the database.

Let's first discuss what is arguably the most important sequence in all of Number Theory and possible all of mathematics as a whole and that's of prime numbers.

1 Prime Numbers - A000040

1.1 Introduction

A **prime number** is a positive integer larger than 1 whose only positive divisors are 1 and the number itself. An integer larger than 1 that is not prime is called **composite**. Note that 1 itself is neither prime nor composite (sometimes it is called a **unit**). The prime numbers begin with

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, ...$$

and so on (see `https://oeis.org/A000040` for more information). Some of you might have already seen these numbers but there's perhaps lots of interesting variations on this sequence that you might not have encountered in the past.

Let's first start with some simple observations and questions

(i) How many prime numbers are even? Why?

(ii) How many prime numbers can you make using some or all of the digits 1, 3, 8?

(iii) Suppose p is a prime number. How many positive divisors does p^3 have?

(iv) Excluding 2 and 5, what numbers must prime numbers end in?

(v) Do you think there are finitely many prime numbers or infinitely many prime numbers? Can you prove your answer?

(vi) Combining the two previous answers, do you think that excluding 2 and 5, there are infinitely many prime numbers that end with the same digit? What can you find by doing an internet search on this result?

In the next two sections, we present some interesting subsets of the prime numbers which have fascinated mathematicians for a very long time. We then conclude with some exercises and other fun sequences of numbers related to primes.

1.2 Mersenne Primes- A000668

There are lots of interesting subsequences of prime numbers but in the interest of time we'll present only a few particularly interesting ones. The first of which is sequence A000668 Mersenne Primes (see `https://primes.utm.edu/mersenne/` or `https://oeis.org/A000668` for more information).

A **Mersenne prime** is a prime number of the form $2^p - 1$ for some prime p. The list begins with:

$$3, 7, 31, 127, 8191, 131071, 524287, 2147483647, 2305843009213693951, ...$$

and the largest known as of 2021 was discovered in 2018 by the GIMPS project (Great Internet Mersenne Prime Search) to be $2^{82,589,933} - 1$.

The history of these numbers is quite interesting. It was widely believed that numbers of the form $2^n - 1$ were always prime for prime numbers n. This was first shown false in 1536 by Hudalricus Regius when they showed that $2^{11} - 1 = 2047$ was not prime.

Exercise: Factor 2047 into a product of prime numbers.

By 1603, another mathematician named Pietro Cataldi showed that $2^{17} - 1$ and $2^{19} - 1$ were both prime. However, Pietro then incorrectly stated that all of $2^{23} - 1$, $2^{29} - 1$, $2^{31} - 1$ and $2^{37} - 1$ were also prime.

Exercise: It turns out that only one of $2^{23} - 1$, $2^{29} - 1$, $2^{31} - 1$ or $2^{37} - 1$ is prime. Which is it? Who proved it was prime first? Using a computer try to factor the other numbers.

Finally, we see the entrance of the French Monk Marin Mersenne after whom the numbers were named. Mersenne stated in the preface to his *Cogitata Physica-Mathematica* (1644) that the numbers $2^n - 1$ were prime for

$$n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127 \text{ and } 257$$

and were composite for all other positive integers $n < 257$. Again this claim was also false (see `https://oeis.org/A000043` for the correct list).

Exercise: Mersenne was wrong about two numbers included in the list above. Using OEIS, determine which two numbers Mersenne thought were prime but are not.

Exercise: Mersenne was wrong about three numbers **not** included in the list above. Using OEIS, determine which three prime numbers Mersenne missed.

Exercise: In the above list for n, the n values share a common property. What is it?

In fact, this is not a coincidence:

Claim: If $2^p - 1$ is a Mersenne prime, then p must itself be a prime number.

Solution: **Note:** If you have not yet learned how to multiply a polynomial by a polynomial, you can learn more about it here: `https://courseware.cemc.uwaterloo.ca/41/133/assignments/1083/0`!

Suppose $2^n - 1$ is a prime number for some integer n. Let's write $n = rs$ for two numbers r and s with $r < s$. Then if think about polynomials, we can write $x^{rs} - 1$ as

$$x^{rs} - 1 = (x^r - 1)(x^{r(s-1)} + x^{r(s-2)} + ... + 1)$$

(This is like how we factor $x^2 - 1$ as $(x-1)(x+1)$ or $x^3 - 1$ as $(x-1)(x^2+x+1)$. Now, setting 2, we see that

$$2^n - 1 = 2^{rs} - 1 = (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} + ... + 1)$$

In other words, we have that $2^r - 1$ is a factor of $2^n - 1$ strictly smaller than $2^n - 1$ (since we know that $r < s \leq n$. Since $2^n - 1$ is prime, it must be that $2^r - 1 = 1$ which shows that $r = 1$. Hence, the number n cannot be expressed as the factor of two numbers where one is not 1. Thus, n is a prime number. ■

Exercise: Using the above idea, show that if $a^n - 1$ is a prime number, then a must be 2 and n must be prime. As a hint notice that $3^4 - 1 = (3-1)(3^3 + 3^2 + 3^1 + 1)$ and $5^3 - 1 = (5-1)(5^2 + 5^1 + 1)$. Can you generalize this pattern?

1.3 Sophie Germain Primes - A005384

Another interesting subsequence of the primes are the **Sophie Germain primes**, primes p where $2p+1$ is also prime (see `https://oeis.org/A005384`). These include:

$$2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233, 239, 251, 281, 293, 359, ...$$

Exercise: The associated prime $2p+1$ is called a **safe prime** (see `https://oeis.org/A005385`). What are the first 5 safe primes?

These primes first came up in the study of Fermat's Last Theorem, the result that the equation $x^n + y^n = z^n$ has no integer solutions assuming $xyz \neq 0$ and $n > 3$. Sophie was the first to prove that this theorem holds whenever n is a Sophie Gernain prime.

Exercise: What is the largest known Sophie Germain prime to date?

Sophie Germain was born in Paris on April 1st, 1776 and spent most if not all of her life in France. She was the daughter of Ambroise-François and Marie-Madeleine Germain. Her father was a wealthy silk merchant and from his wealth had amassed a sizeable library. Germain began her interest in mathematics around 1789 (the heart of the French Revolution) when she was 13 reading books in her father's library. She read a story about the death of Archimedes in Montucla' *Histoire des mathématiques* inspiring her to study mathematics. Guglielmo Libri Carucci dalla Sommaja (one of her biographers) writes that Germain would wake up in the middle of the night to do mathematics. Her parents removed her fire, clothes and candles from her room. Undeterred, she awoke under dim lamp light to do mathematics (even with a frozen ink well!) The French Revolution forced her to stay home and as a consequence she spent much time in her father's library reading and studying mathematics.

For Interest: Look up the French Revolution. What were some of the causes of the revolution? How did this influence Sophie Germain's life?

2 Exercises

2.1 Prime Questions

(i) A **twin prime** is a prime p where $p+2$ is also prime (see `https://oeis.org/A001359`). How many twin primes are there less than 100? What are they? Can you find one above 100?

(ii) **Goldbach's Conjecture** states that every even number larger than 2 can be expressed as the sum of two primes. For example, $4 = 2 + 4$, $6 = 3 + 3$ and $8 = 3 + 5$ (see `https://oeis.org/A002372`). Write the number 30 as the sum of two prime numbers. How many ways can you do this? (Assume rearranging the terms in the sum counts as the same way).

(iii) A **perfect number** is a number that is equal to the sum of its positive proper divisors (the divisors not equal to the number itself). For example 6 is perfect since $6 = 1 + 2 + 3$ and 6 is a perfect number.

1. Recall that $7 = 2^3 - 1$ is a Mersenne prime. Show that $2^2(2^3 - 1) = 28$ is a perfect number.
2. (Challenging!) More generally, show that if $2^p - 1$ is a Mersenne prime then $2^{p-1}(2^p - 1)$ is a perfect number. It might help to use the fact that $1 + 2 + ... + 2^{p-1} = 2^p - 1$
3. (Very Challenging!) Conversely, show that every even perfect number is of the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is a Mersenne prime. (Hint: Write the even perfect number as $2^k n$ where n is odd).

(iv) Sophie Germain also has an identity named after her, namely

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$$

Prove that $3^{44} + 4^{29}$ is composite using the identity.

2.2 Slime Numbers (With thanks to Henri Picciotto) - A166504

Cite: `www.MathEducationPage.org`

Define a **slicing** as a number as collections of slices of consecutive digits where each digit belongs to one and only one such slice. For example, we can slice a number like 123 in many ways:

$$\{123\}, \{1, 23\}, \{12, 3\}, \{1, 2, 3\}$$

Further, we say that a number is **slime** if one of the above slices consists only of primes. So above 123 is not slime since all such slices contain a non-prime but a number like 1705 is slime since the slice $\{17, 05\}$ (we drop the leading 0) consists only of prime numbers. Note that every prime number is a slime number by taking the slice with the number itself in it.

(i) Slice the numbers 1234 and 56789 in all possible ways.

(ii) Find the first three examples of composite numbers that are slime.

(iii) Find the first three even slimes

(iv) Find the first three slime squares

(v) Find the first three slime cubes

(vi) Are there any fourth powers that are slime? Find one if there is!

(vii) Find the first three pairs of slime numbers that are consecutive integers.

(viii) Find the first three triples of slime numbers that are consecutive integers.

(ix) Prove that there are infinitely slime numbers that are not prime.

(x) Find the smallest number that is slime in more than one way. (In other words, it can be sliced into two different sequences of primes.)

(xi) (Challenging!) A number is a super-slime if you get a sequence of primes no matter how you slice it. For example, 53 is a super-slime since $\{53\}$ and $\{5, 3\}$ are slices consisting only of primes. Prove that there are only a finite number of super-slimes and find them all! It might help to use a list of prime numbers to help with larger ones.