OEIS Math Circles Part 1

In this two part series, we're going to look at a few of my favourite sequences in great depth. Our goals of this learning activity are the following:

(i) An exploration into prime numbers and some interesting sequences relating to prime numbers

(ii) An exploration into recursive sequences, namely the Fibonacci Numbers, Lucas Numbers and others

(iii) Explore `www.oeis.org` and find some new sequences!

Relating to the last point, we will frequently make references to `www.oeis.org` and periodically we will use the numbering scheme used there to refer to sequences. As an example, go to the above website and type in **A000001** for the first sequence in the database.

Let's first discuss what is arguably the most important sequence in all of Number Theory and possible all of mathematics as a whole and that's of prime numbers.

1 Prime Numbers - A000040

1.1 Introduction

A **prime number** is a positive integer larger than 1 whose only positive divisors are 1 and the number itself. An integer larger than 1 that is not prime is called **composite**. Note that 1 itself is neither prime nor composite (sometimes it is called a **unit**). The prime numbers begin with

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, ...$$

and so on (see `https://oeis.org/A000040` for more information). Some of you might have already seen these numbers but there's perhaps lots of interesting variations on this sequence that you might not have encountered in the past.

Let's first start with some simple observations and questions

(i) How many prime numbers are even? Why?

> **Solution**
>
> There is only one even prime number and that is 2. Notice that all even numbers larger than 2 must be divisible by to and so must be composite.

(ii) How many prime numbers can you make using some or all of the digits 1, 3, 8?

> **Solution**
>
> 3, 13, 31, 83. Notice that 381 and 183 are divisible by 3. Interesting any number where the sum of the digits is divisible by 3 is also divisible by 3!

(iii) Suppose p is a prime number. How many positive divisors does p^3 have?

Solution

It has 4 positive divisors, namely 1, p, p^2 and p^3.

(iv) Excluding 2 and 5, what numbers must prime numbers end in?

Solution

Prime numbers must end with an odd number and not 5 (numbers that end in 5 are divisible by 5!). This means they must end with the digits 1, 3, 7 or 9. See these sequences:

- `https://oeis.org/A030430`
- `https://oeis.org/A030431`
- `https://oeis.org/A030432`
- `https://oeis.org/A030433`

(v) Do you think there are finitely many prime numbers or infinitely many prime numbers? Can you prove your answer?

Solution

There are infinitely many prime numbers! The idea uses this fact:

Fundamental Theorem of Arithmetic Every integer larger than 1 is either prime or is expressible as a product of primes uniquely up to reordering of the factors.

The above result is fairly easy to understand - if a number isn't prime, then it can be written as a product of two numbers rs with neither r nor s being 1. Then either r and s are prime or we repeat the process until they are. Since the numbers we get are strictly decreasing and bounded below by 1, this process must stop. There is still the question of uniqueness but we'll save that for another day. You can formally prove this by using induction but we forgo that for now.

Suppose we started to make a list of prime numbers, say $p_1, p_2, ..., p_k$. Construct the number $N = p_1 p_2 ... p_k + 1$. Now this number is the product of primes by the above result and it isn't divisible by one of the primes we already had (because if say p_j divides N then since p_j divides $p_1 p_2 ... p_k$, it must divide their difference which is 1. This is not possible so indeed the prime factors of N cannot be one of the k we already listed). In this way we can create a new prime number given any finite list and so there must be infinitely many prime numbers.

(vi) Combining the two previous answers, do you think that excluding 2 and 5, there are infinitely many prime numbers that end with the same digit? What can you find by doing an internet search on this result?

Solution

An incredibly interesting theorem is Dirichlet's Theorem on Primes in Arithmetic Progressions. It states that numbers of the form $10k + 1$ or $10k + 3$ or $10k + 7$ or $10k + 9$ must each include infinitely many prime numbers (and in general numbers of the form $ak + b$ must contain infinitely many prime numbers so long as a and b don't share a prime factor!)

In the next two sections, we present some interesting subsets of the prime numbers which have fascinated mathematicians for a very long time. We then conclude with some exercises and other fun sequences of numbers related to primes.

1.2 Mersenne Primes- A000668

There are lots of interesting subsequences of prime numbers but in the interest of time we'll present only a few particularly interesting ones. The first of which is sequence A000668 Mersenne Primes (see `https://primes.utm.edu/mersenne/` or `https://oeis.org/A000668` for more information).

A **Mersenne prime** is a prime number of the form $2^p - 1$ for some prime p. The list begins with:

$$3, 7, 31, 127, 8191, 131071, 524287, 2147483647, 2305843009213693951, \ldots$$

and the largest known as of 2021 was discovered in 2018 by the GIMPS project (Great Internet Mersenne Prime Search) to be $2^{82,589,933} - 1$.

The history of these numbers is quite interesting. It was widely believed that numbers of the form $2^n - 1$ were always prime for prime numbers n. This was first shown false in 1536 by Hudalricus Regius when they showed that $2^{11} - 1 = 2047$ was not prime.

Exercise: Factor 2047 into a product of prime numbers.

Solution

Note that $2047 = 23 \cdot 89$

By 1603, another mathematician named Pietro Cataldi showed that $2^{17}-1$ and $2^{19}-1$ were both prime. However, Pietro then incorrectly stated that all of $2^{23}-1$, $2^{29}-1$, $2^{31}-1$ and $2^{37} - 1$ were also prime.

Exercise: It turns out that only one of $2^{23} - 1$, $2^{29} - 1$, $2^{31} - 1$ or $2^{37} - 1$ is prime. Which is it? Who proved it was prime first? Using a computer try to factor the other numbers.

Solution

- $2^{23} - 1 = 47 \cdot 178481$ as shown by Fermat in 1640.
- $2^{29} - 1 = 233 \cdot 1103 \cdot 2089$ as shown by Euler in 1738.
- $2^{31} - 1$ is prime and was first proven to be prime by Euler in 1772.
- $2^{37} - 1 = 223 \cdot 616318177$ as shown by Fermat in 1640.

Finally, we see the entrance of the French Monk Marin Mersenne after whom the numbers were named. Mersenne stated in the preface to his *Cogitata Physica-Mathematica* (1644) that the numbers $2^n - 1$ were prime for

$$n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127 \text{ and } 257$$

and were composite for all other positive integers $n < 257$. Again this claim was also false (see `https://oeis.org/A000043` for the correct list).

Exercise: Mersenne was wrong about two numbers included in the list above. Using OEIS, determine which two numbers Mersenne thought were prime but are not.

Solution

The numbers $2^{67} - 1$ and $2^{257} - 1$ are not prime.

Exercise: Mersenne was wrong about three numbers **not** included in the list above. Using OEIS, determine which three prime numbers Mersenne missed.

Solution

The numbers $2^{61} - 1$, $2^{89} - 1$ and $2^{107} - 1$ are all Mersenne primes.

Exercise: In the above list for n, the n values share a common property. What is it?

Solution

All the numbers are prime!

In fact, this is not a coincidence:

Claim: If $2^p - 1$ is a Mersenne prime, then p must itself be a prime number.

Solution: **Note:** If you have not yet learned how to multiply a polynomial by a polynomial, you can learn more about it here: `https://courseware.cemc.uwaterloo.ca/41/133/assignments/1083/0`!

Suppose $2^n - 1$ is a prime number for some integer n. Let's write $n = rs$ for two numbers r and s with $r < s$. Then if think about polynomials, we can write $x^{rs} - 1$ as

$$x^{rs} - 1 = (x^r - 1)(x^{s(r-1)} + x^{s(r-2)} + ... + 1)$$

(This is like how we factor $x^2 - 1$ as $(x - 1)(x + 1)$ or $x^3 - 1$ as $(x - 1)(x^2 + x + 1)$. Now, setting 2, we see that

$$2^n - 1 = 2^{rs} - 1 = (2^r - 1)(2^{s(r-1)} + 2^{s(r-2)} + ... + 1)$$

In other words, we have that $2^r - 1$ is a factor of $2^n - 1$ strictly smaller than $2^n - 1$ (since we know that $r < s \leq n$. Since $2^n - 1$ is prime, it must be that $2^r - 1 = 1$ which shows that $r = 1$. Hence, the number n cannot be expressed as the factor of two numbers where one is not 1. Thus, n is a prime number. ■

Exercise: Using the above idea, show that if $a^n - 1$ is a prime number, then a must be 2 and n must be prime. As a hint notice that $3^4 - 1 = (3 - 1)(3^3 + 3^2 + 3^1 + 1)$ and $5^3 - 1 = (5 - 1)(5^2 + 5^1 + 1)$. Can you generalize this pattern?

Solution

Notice that $a-1$ must divide a^n-1 since $(a-1)(a^{n-1}+a^{n-2}+...+1)=a^n-1$. Thus, since $a-1$ would be a proper divisor, a number of the form a^n-1 could only be prime if $a-1=1$ or in other words $a=2$. The proof that n must be prime is thus above. ■

1.3 Sophie Germain Primes - A005384

Another interesting subsequence of the primes are the **Sophie Germain primes**, primes p where $2p+1$ is also prime (see `https://oeis.org/A005384`). These include:

$$2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233, 239, 251, 281, 293, 359, ...$$

Exercise: The associated prime $2p+1$ is called a **safe prime** (see `https://oeis.org/A005385`). What are the first 5 safe primes?

Solution

2(2) + 1 = 5, 2(3) + 1 =7, 2(5)+1 = 11, 2(11) + 1 = 23, 2(23) + 1 = 47

These primes first came up in the study of Fermat's Last Theorem, the result that the equation $x^n+y^n=z^n$ has no integer solutions assuming $xyz \neq 0$ and $n>3$. Sophie was the first to prove that this theorem holds whenever n is a Sophie Gernain prime.

Exercise: What is the largest known Sophie Germain prime to date?

Solution

The largest is $2618163402417 \cdot 2^{1290000} - 1$ discovered in 2016.

Sophie Germain was born in Paris on April 1st, 1776 and spent most if not all of her life in France. She was the daughter of Ambroise-François and Marie-Madeleine Germain. Her father was a wealthy silk merchant and from his wealth had amassed a sizeable library. Germain began her interest in mathematics around 1789 (the heart of the French Revolution) when she was 13 reading books in her father's library. She read a story about the death of Archimedes in Montucla' *Histoire des mathématiques* inspiring her to study mathematics. Guglielmo Libri Carucci dalla Sommaja (one of her biographers) writes that Germain would wake up in the middle of the night to do mathematics. Her parents removed her fire, clothes and candles from her room. Undeterred, she awoke under dim lamp light to do mathematics (even with a frozen ink well!) The French Revolution forced her to stay home and as a consequence she spent much time in her father's library reading and studying mathematics.

For Interest: Look up the French Revolution. What were some of the causes of the revolution? How did this influence Sophie Germain's life?

2 Exercises

2.1 Prime Questions

(i) A **twin prime** is a prime p where $p + 2$ is also prime (see `https://oeis.org/A001359`). How many twin primes are there less than 100? What are they? Can you find one above 100?

Solution

Twin primes less than 100 are:

$$3, 5, 11, 17, 29, 41, 59, 71$$

Corresponding the the primes $p + 2$:

$$5, 7, 13, 19, 31, 43, 61, 73$$

Note that 101 and 103 are both prime so 101 is a twin prime larger than 100.

(ii) **Goldbach's Conjecture** states that every even number larger than 2 can be expressed as the sum of two primes. For example, $4 = 2 + 4$, $6 = 3 + 3$ and $8 = 3 + 5$ (see `https://oeis.org/A002372`). Write the number 30 as the sum of two prime numbers. How many ways can you do this? (Assume rearranging the terms in the sum counts as the same way).

Solution

Note that

$$30 = 7 + 23 = 11 + 19 = 13 + 17$$

and these are the only ways to write 30 as the sum of two prime numbers up to reordering. Thus there are 3 such ways.

(iii) A **perfect number** is a number that is equal to the sum of its positive proper divisors (the divisors not equal to the number itself). For example 6 is perfect since $6 = 1 + 2 + 3$ and 6 is a perfect number.

1. Recall that $7 = 2^3 - 1$ is a Mersenne prime. Show that $2^2(2^3 - 1) = 28$ is a perfect number.

   Solution

   Note that the positive proper divisors of 28 are

   $$1, 2, 4, 7, 14$$

   and the sum is $1 + 2 + 4 + 7 + 14 = 28$ so 28 is perfect.

2. (Challenging!) More generally, show that if $2^p - 1$ is a Mersenne prime then $2^{p-1}(2^p - 1)$ is a perfect number. It might help to use the fact that $1 + 2 + ... + 2^{p-1} = 2^p - 1$

Solution

Let $q = 2^p - 1$ for simplicity. The positive proper divisors of this number are are:

$$1, q, 2, 2q, 2^2, 2^2q, ...2^{p-2}, 2^{p-2}q, 2^{p-1}$$

We split these up into numbers that have a factor of q and those that don't and add these up:

$$\begin{aligned}(1 + 2{+}2^2 + ... + 2^{p-2} + 2^{p-1}) &+ q(1 + 2 + 2^2 + ... + 2^{p-2}) \\ &= (2^p - 1) + (2^p - 1)(2^{p-1} - 1) \\ &= (2^p - 1)(1 + 2^{p-1} - 1) \\ &= 2^{p-1}(2^p - 1)\end{aligned}$$

proving the number is perfect.

3. (Very Challenging!) Conversely, show that every even perfect number is of the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is a Mersenne prime. (Hint: Write the even perfect number as $2^k n$ where n is odd).

Solution

Let N be an even perfect number and write it as $2^{m-1}n$ where n is odd and m is the number such that $m+1$ is the number of factors of 2 of N. Then let $d_1, ...d_k$ be the odd factors of n with $d_k = n$ being the largest factor so that $2^{m-1}d_k = N$. Then the sum of all the factors is

$$\begin{aligned}N = 2^{m-1}n &= \sum_{i=1}^{k} d_i + 2\sum_{i=1}^{k-1} d_i + 2^2\sum_{i=1}^{k-1} d_i + ... + 2^{m-1}\sum_{i=1}^{k-1} d_i - 2^{m-1}n \\ &= (1 + 2 + ... + 2^{m-1})\sum_{i=1}^{k} d_i - 2^{m-1}n \\ &= (2^m - 1)\sum_{i=1}^{k} d_i - 2^{m-1}n\end{aligned}$$

(Notice that the second sum above is missing $2d_k$ since $2d_k = n$). Rearranging gives

$$2^m n = (2^m - 1)\sum_{i=1}^{k} d_i$$

So now, $2^m - 1$ divides n and we can write $n = (2^m - 1)M$ for some integer M (so rearranging gives $n + m = 2^m M$). Substituting this in and simplifying gives

$$2^m M = \sum_{i=1}^{k} d_i$$

Now, the right hand side is the sum of all odd divisors of n (and n is odd) and we have that M is such an odd divisor and of course n is and so

$$2^m M = \sum_{i=1}^{k} d_i \geq n + M = 2^m M$$

and so the only divisors of n are n and $M < n$ (so M is 1). Hence, $n = 2^m - 1$ and indeed must be prime since there are only two divisors. Thus, $N = 2^{m-1}n = 2^{m-1}(2^m - 1)$ with $2^m - 1$ prime as required.

(iv) Sophie Germain also has an identity named after her, namely

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$$

Prove that $3^{44} + 4^{29}$ is composite using the identity.

Solution

Note that

$$3^{44}+4^{29} = (3^{11})^4+4(47)^4 = (3^{22}+2(3^{11})(4^7)+2(4^{14}))(3^{22}-2(3^{11})(4^7)+2(4^{14}))$$

and both the factors are larger than 1 hence the number is composite.

2.2 Slime Numbers (With thanks to Henri Picciotto) - A166504

Cite: `www.MathEducationPage.org`

Define a **slicing** as a number as collections of slices of consecutive digits where each digit belongs to one and only one such slice. For example, we can slice a number like 123 in many ways:

$$\{123\}, \{1, 23\}, \{12, 3\}, \{1, 2, 3\}$$

Further, we say that a number is **slime** if one of the above slices consists only of primes. So above 123 is not slime since all such slices contain a non-prime but a number like 1705 is slime since the slice $\{17, 05\}$ (we drop the leading 0) consists only of prime numbers. Note that every prime number is a slime number by taking the slice with the number itself in it.

(i) Slice the numbers 1234 and 56789 in all possible ways.

Solution

For 1234, we have the slices

$$\{1234\}, \{1, 234\}, \{1, 2, 34\}, \{1, 23, 4\}, \{1, 2, 3, 4\}, \{12, 34\}, \{12, 3, 4\}, \{123, 4\}$$

for a total of 8 slices. As for 56789:

$$\{56789\}, \{5, 6789\}, \{5, 6,789\}, \{5, 67, 89\}, \{5, 6, 7, 89\},$$
$$\{5, 678, 9\}, \{5, 6, 78, 9\},\{5, 678, 9\}, \{5, 6, 7, 8, 9\},$$
$$\{56, 789\}, \{56, 7, 89\}, \{56, 78, 9\}, \{56, 7,8, 9\}, \{567, 89\}, \{567, 8, 9\}, \{5678, 9\}$$

for a total of 16 slices. Can you guess how many slices there would be for a 6 digit number?

(ii) Find the first three examples of composite numbers that are slime.

Solution

22, 25, 27 are the first three composite slime numbers.

(iii) Find the first three even slimes

Solution

2, 22, 32 are the first three even slime numbers.

(iv) Find the first three slime squares

Solution

25, 225, 289 are the first three slime squares.

(v) Find the first three slime cubes

Solution

27, 343, 729 are the first slime cubes.

(vi) Are there any fourth powers that are slime? Find one if there is!

Solution

There is one! $7^4 = 2401$ which is slime since it can be sliced as $\{2, 401\}$ and 401 is prime!

(vii) Find the first three pairs of slime numbers that are consecutive integers.

Solution

- 2, 3 is a consecutive pair of slime numbers
- 22, 23 is a consecutive pair of slime numbers
- 31, 32 is a consecutive pair of slime numbers.

(viii) Find the first three triples of slime numbers that are consecutive integers.

Solution

- 31, 32, 33 is a consecutive triple of slimes.
- 71, 72, 73 is a consecutive triple of slimes.
- 131, 132, 133 is a consecutive triple of slimes (note that 131 is prime).

Exercise: Why are these the first three?

(ix) Prove that there are infinitely slime numbers that are not prime.

Solution

Notice that the repdigit number 22222...222 is always slime and not prime (except for 2 of course!) Hence there are infinitely many non-prime slime numbers.

(x) Find the smallest number that is slime in more than one way. (In other words, it can be sliced into two different sequences of primes.)

Solution

Notice that 23 is a prime number and can be sliced as {2, 3} which also shows the number is slime.

(xi) (Challenging!) A number is a super-slime if you get a sequence of primes no matter how you slice it. For example, 53 is a super-slime since $\{53\}$ and $\{5, 3\}$ are slices consisting only of primes. Prove that there are only a finite number of super-slimes and find them all! It might help to use a list of prime numbers to help with larger ones.

Solution

First, note that 2, 3, 5, 7 are all super-slimes. Then, the numbers 23, 37, 53, 73 are all two-digit super-slimes. To get further super-slimes, we need to take smaller ones and add a digit to the end (since if you remove the last digit, the remaining number must still be a super-slime!) The last digit can only be one of 3 or 7 since if you append either a 1, 2, 4, 5, 6, 8 or 9 then either the number itself is even or slicing the last number gives you a non-prime number. So we take the two digit super-slime numbers and try to append each possible digit. Further, the last two digits cannot be the same since otherwise the last two digits are divisible by 11 which eliminates the slices 233, 377, 533 and 733.

- 237 is not a super-slime ({237} is not a valid slice since 237 is divisible by 3)
- 373 is a super-slime (the slices {3, 7, 3}, {373}, {3, 73} and {37, 3} are all slices of primes)
- 537 is not a super-slime ({537} is not a valid slice since 537 is divisible by 3)
- 737 is not a super-slime ({737} is not a valid slice since 737 is divisible by 11)

This leaves one candidate, namely 373. We look at 3737 but clearly this number is divisible by 37 and so is not a super-slime. Hence the complete list of super-slimes is

$$2, 3, 5, 7, 23, 37, 53, 73, 373.$$

See `https://oeis.org/A085823`!