



## Grade 6 Math Circles

October 13, 2021

### Divisibility, Factors, and the GCF

The English word **divide** means to split a quantity into parts. For example, you can divide a pile of coins into three different, smaller piles, or divide a pie into 8 slices. In Mathematics, when we say that a positive integer  $x$  divides another number  $y$ , it means that  $y$  can be split into  $x$  **equal** parts without a remainder. (In this lesson, we will only be working with positive integers for simplicity's sake).

We will use the statement below to go over some important terminology.

$$12 \div 3 = 4 \text{ R } 0$$

In this example, 12 is the *dividend*, 3 is the *divisor*, 4 is the *quotient*, and 0 is the *remainder*. Since we can divide 12 into 3 groups of 4 with nothing left over, we can say that 3 **divides** 12, and 12 **is divisible** by 3.

When a number  $x$  divides another number  $y$ , we say that  $x$  is a **factor** of  $y$ . Hence, 3 is a factor of 12.

#### Stop and Think

Is 4 a factor of 12? How many factors of 12 are there?

### Prime Numbers

A **prime number** is a positive integer that has exactly 2 distinct factors: 1 and itself. In contrast, a **composite number** is a positive integer with more than 2 factors. Every integer greater than 1 is either prime or composite.

#### Stop and Think

Is 1 prime, composite, or neither?

**Exercise 1**

Classify all integers greater than 1 and less than 20 as either a prime or a composite number.

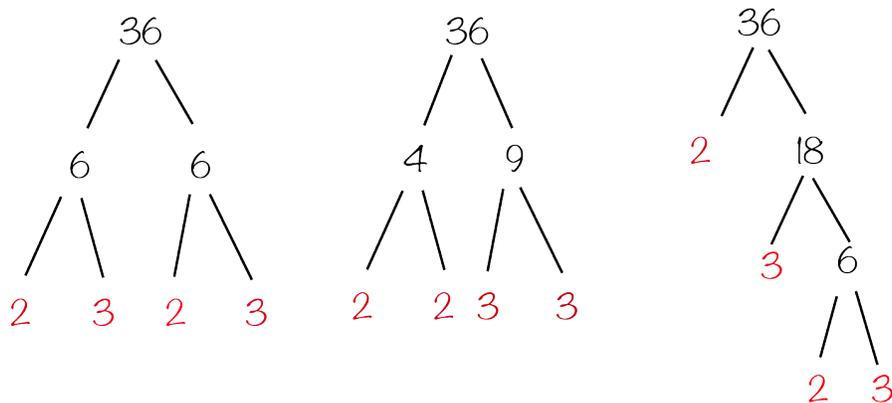
The **Fundamental Theorem of Arithmetic** is a very important and influential theorem in mathematics. The theorem states that every integer greater than 1 can be uniquely written as a product of prime numbers.

For example, the number 30 can be written as  $2 \times 3 \times 5$ . Similarly, 100 can be written as  $2 \times 2 \times 5 \times 5$ . This is called the **prime factorization** of a number.

We can find the prime factorization of a number using a method called the *factor tree*. Suppose we want to find the prime factorization of the number  $x$ . We can start constructing an “upside down” tree starting with  $x$  as the root. First, break the number  $x$  into the product of two numbers. Then for each of those two numbers, break it up again into the product of two more numbers. Continue each “branch” of the tree until you arrive at a prime number at the end of each branch. At the end of the algorithm, simply take the “leaves” of the tree as the prime factorization of  $x$ .

There can be more than one tree for the same number, but the algorithm will always give you the correct answer.

For example, here are some factor trees of 36. The prime factorization of 36 is  $2 \times 2 \times 3 \times 3$ . Notice that the order of the 2’s and 3’s does not matter, since we are dealing with multiplication.

**Exercise 2**

Use factor trees to find the prime factorization of 30 and 48.



## All Factors of a Number

Now that we know how to find all the prime factors of a number, how do we find all the (not necessarily prime) factors? Before we start finding its factors, it will be helpful to know how many factors a number has, so we won't miss any during our computation.

### The Factors Formula

The Factors Formula makes use of a number's prime factorization. In order to use it, we need to express the prime factorization in exponential notation. For example,  $8 = 2 \times 2 \times 2 = 2^3$ . Here, 2 is called the *base* and 3 is the *exponent*. This expression is often stated as "2 to the power of 3" or "2 cubed", which means that 2 is multiplied by itself 3 times. Similarly,  $9 = 3 \times 3 = 3^2$ .

If a number is not multiplied by anything, its exponent is 1. Even though we typically do not write the number 2 as  $2^1$ , they are equivalent.

Using exponents, we can shorten a number's prime factorization. For example, we know the prime factorization of 100 is  $2 \times 2 \times 5 \times 5$ . In exponential notation, it is  $2^2 \times 5^2$ . Similarly, we can write the prime factorization of 30 as  $2^1 \times 3^1 \times 5^1$ .

The Factors Formula states that given the prime factorization of a number in exponential notation, the number of factors the number has is the product of (each exponent + 1). In other words, if the prime factorization of a number is

$$p_0^{a_0} \times p_1^{a_1} \times \dots \times p_n^{a_n} ,$$

then the number of factors it has is

$$(a_0 + 1) \times (a_1 + 1) \times \dots \times (a_n + 1)$$

For example, if we wanted to find the number of factors of 20, we can first find its prime factorization using a factor tree, and obtain  $20 = 2 \times 2 \times 5 = 2^2 \times 5^1$ , so the number of factors of 20 is  $(2+1) \times (1+1) = 3 \times 2 = 6$ .

#### Exercise 3

How many factors does 50 have?



## Finding All Factors of a Number

We now know that there are 6 factors of 20, so let us find those 6 factors. We can use a method which we will refer to as the “train method” to accomplish this.

Imagine a train split into many different cars. We will represent the train with a long rectangle and lines to divide it into cars. Our goal is to find “factor pairs” of 20, which are two numbers that multiply together to get 20, and put each factor in the first and last empty cars of the train, until the train is filled with every factor.

We know that every prime and composite number has at least two factors - 1 and itself - so we can place 1 and 20 into the first and last car of the train.



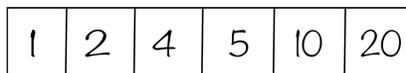
Since we have 1 as a factor, we will count up from 1. Since 2 is also a factor of 20 (and by mental math we know that  $2 \times 10 = 20$ ), we have another pair of factors: 2 and 10.



Note that 3 does not divide 20 (since  $20 \div 3 = 6$  with a remainder of 2), so 3 is not a factor of 20. However, 4 is a factor of 20. Since  $4 \times 5 = 20$ , we can put another factor pair into the train.



There are no integers between 4 and 5, so our algorithm stops at this step, and the train is now full. Therefore, 1, 2, 4, 5, 10, and 20 are all the factors of 20.



One important rule for finding all factors of a number is that every factor will only appear once on the factor train. Therefore, if a factor pair consists of two identical numbers (i.e.  $25 = 5 \times 5$ ), 5 will only appear once on the factor train (i.e. the factors of 25 are 1, 5, and 25).

**Exercise 4**

Find all the factors of 50 using the factor train.

## The Greatest Common Factor

From our computation and the exercise above, we know that

- the factors of 20 are 1, 2, 4, 5, 10, and 20
- the factors of 50 are 1, 2, 5, 10, 25, and 50

Notice that some numbers appear in both lists. These numbers are called the **common factors** of 20 and 50. The largest of these numbers is called the **greatest common factor** (GCF). Another name for the GCF is the greatest common divisor (GCD), but we will use GCF in this lesson. In the example above, the GCF of 20 and 50 is 10.

**Stop and Think**

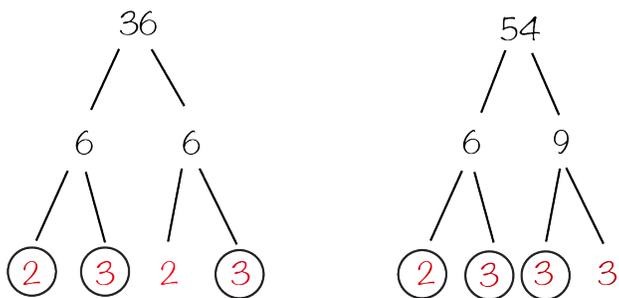
What is the GCF of two different prime numbers?

The GCF is used in many branches of mathematics, particularly in number theory. It is also used for important cryptography algorithms such as the RSA algorithm. In these algorithms, it is often necessary to compute the GCF of two very large numbers very quickly. If we compute every single factor of each of the two large numbers, find the list of common factors, and pick out the GCF, the process would take very long. Thankfully, there are faster methods, and we will discuss two of them in this lesson.

## Finding the GCF Using Factor Trees

Now that we know how to make factor trees, we can also use factor trees to find the GCF of two numbers. First, write down the factor tree of each number. Then, find the prime factors that are common in the two trees. Finally, multiply the common prime factors together to obtain the GCF.

Here is an example using the factor trees of 36 and 54.



The prime factors shared by 36 and 54 are 2, 3, and 3. When we multiply them together, we see that the GCF of 36 and 54 is  $2 \times 3 \times 3 = 18$ .

### Finding the GCF Using the Euclidean Algorithm

The Euclidean Algorithm is an efficient way to find the GCF of two numbers. This method uses long division, so make sure you are comfortable with it. Try the exercise below if you need a refresher.

**Exercise 5**  
Use long division to compute  $167 \div 3$ , and state the quotient and remainder.

Let us illustrate the Euclidean Algorithm using an example. Suppose we want to find the GCF of 40 and 56.

Step 1: Divide the larger number by the smaller number. In other words, let the larger number be the dividend and the smaller number be the divisor and perform long division to obtain a quotient and (possibly) a remainder.

$$\begin{array}{r} 1 \\ 40 \overline{)56} \\ \underline{40} \\ 16 \end{array}$$

Step 2: Divide the previous divisor by the remainder. In this case, 40 is the new dividend and 16 is the new divisor.

$$\begin{array}{r} 2 \\ 16 \overline{)40} \\ \underline{32} \\ 8 \end{array}$$



Step 3: Repeat step 2 again. Here, 16 is the new dividend and 8 is the new divisor.

$$\begin{array}{r} 2 \\ 8 \overline{)16} \\ \underline{16} \\ 0 \end{array}$$

At any point of the algorithm, if you see a remainder of 0, then the algorithm terminates. The GCF of the original two numbers is the last non-zero remainder (or the current divisor). In this case, the GCF of 40 and 56 is 8.

If the remainder is not 0, keep repeating the second step by using the previous divisor as the dividend and the previous remainder as the new divisor. This algorithm will always terminate. (Why?)

#### Exercise 6

Find the GCF of 40 and 56 using factor trees. Did you obtain the same answer?

#### Exercise 7

Find the GCF of 36 and 54 using the Euclidean Algorithm. Did you obtain the same answer as above, when we used the factor tree method?

#### Stop and Think

Which of the two methods do you like better? If we want to find the GCF of two extremely large numbers, which method would you prefer? What about the GCF of three numbers?