Grade 6 Math Circles

October 6, 2021

Irrational Numbers - Solutions

Exercise Solutions

Exercise 1

Identify the irrational number(s) from the options below.

(a) $\sqrt{8}$ (b) 2021.1006 (c) $\frac{-79}{1084}$ (d) $\sqrt{9}$ (e) $\frac{0}{\sqrt{2}}$

Exercise 1 Solution

(a) is the only irrational number.

Exercise 2

Find the reciprocal of the following numbers. Express your answer as an integer or fraction.

(a) $\frac{1}{5}$ (b) $\frac{17}{36}$ (c) $\frac{-1}{10}$ (d) 36.08 (e) 0.75681

Exercise 2 Solution

(a) 5 (b) $\frac{36}{17}$ (c) -10 (d) $\frac{100}{3608}$ (e) $\frac{1}{0.75681}$

Exercise 3

Continue the algorithm one more step and find a more accurate rational approximation for π.

Exercise 3 Solution

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1.003462085...}}} \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113}$$

Exercise 4

Convert 4.357 into an improper fraction (an improper fraction is a fraction with the numerator bigger than the denominator) by using its continued fraction above. Can you convert 4.357 to an improper fraction in a different way?

Exercise 4 Solution

Given the continued fraction expansion of 4.357, we can simplify it starting from the innermost denominator.

$$\begin{aligned}
4.357 &= 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{35 + \cfrac{1}{2}}}}} \\
&= 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{2}{71}}}} \\
&= 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{71}{286}}} \\
&= 4 + \cfrac{1}{2 + \cfrac{286}{357}} \\
&= 4 + \frac{357}{1000} \\
&= \frac{4357}{1000}
\end{aligned}$$

You can also express 4.357 as a mixed number first, and then as an improper fraction.
i.e. $4.357 = 4\frac{357}{1000} = \frac{4357}{1000}$.

Exercise 5: Putting it all together

Compute the continued fraction representation of $\sqrt{2}$ up to the third step, its rational approximation to $\sqrt{2}$ at that step (as an improper fraction), and the 3rd convergent to $\sqrt{2}$ (using the simplified notation). Do you think this is a good approximation for $\sqrt{2}$?

Exercise 5 Solution

The continued fraction of $\sqrt{2}$ up to the third step is $1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = [1; 2, 2, 2] = \frac{17}{12}$.

Problem Set Solutions

1. For each of the following numbers, state if they are rational, irrational, or neither. Explain the reasoning behind your choice.

   (a) 7 (b) $\frac{\pi}{2}$ (c) $17.181818...$ (d) $\sqrt{16}$ (e) 0

   Solution:

   (a) Rational (b) Irrational (c) Rational (d) Rational (e) Rational

2. State 3 differences between irrational numbers and rational numbers.

   Solution: Sample Answer:
   - Irrational numbers have infinite, non-repeating decimal expansions, while rational numbers have finite or repeating decimal expansions.
   - The continued fraction expansion of irrational numbers are infinite, while the continued fraction expansions for rationals are finite.
   - Rational numbers can be written in the form of a fraction with an integer numerator and a positive integer denominator, while irrational numbers cannot.

3. Compute the following. Express your answer as a whole number or a fraction.

(a) $3 + \frac{2}{5}$ (b) $\frac{1}{\left(\frac{3}{11}\right)}$ (c) $\frac{1}{\left(\frac{1}{12345}\right)}$ (d) $2 + \frac{1}{6 + \frac{1}{3}}$

Solution:

(a) $\frac{17}{5}$ (b) $\frac{11}{3}$ (c) 12345 (d) $\frac{41}{19}$

4. A number's continued fraction expansion is shown below.

$$3 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}}}}$$

(a) Is this number rational or irrational?

Solution: It is a rational number.

(b) What is this number, in improper fraction form?

Solution: $\frac{418}{111}$

(c) What is this number, in decimal form?

Solution: $3.765765765765... = 3.\overline{765}$

5. Find the 5^{th} convergent rational approximation for the golden ratio ϕ. What do you notice? Can you predict what the 10^{th} convergent rational approximation is? You may express your answer in the shortened notation.

> *Solution*: The 5^{th} convergent is $\phi \approx [1;1,1,1,1,1]$. We notice that all the denominators are 1 as we continue the fraction. Thus, we predict that the 10^{th} convergent is $\phi \approx [1;1,1,1,1,1,1,1,1,1,1]$. (Look it up! Are we correct?)

6. Research some of the most famous irrational numbers: π, e, and the golden ratio ϕ. How are these numbers defined? How were they discovered? What are some interesting properties? Can you find how they are used in the real world?

> *Solution*: This is an open-ended question that you should spend some time answering. Feel free to share any interesting things you find with us on Piazza or during the live session!

7. Consider two rational numbers a and b. If the continued fraction representation of the two numbers are

$$a = [a_0; a_1, a_2, a_3, \ldots, a_n]$$
$$b = [0; a_0, a_1, a_2, a_3, \ldots, a_n],$$

what can you say about the relationship of the two numbers?

Solution: The two numbers are reciprocals of each other.
This problem is best solved using concrete examples. Let $a = [1; 2, 3]$ and $b = [0; 1, 2, 3]$.
Thus,

$$a = 1 + \cfrac{1}{2 + \cfrac{1}{3}} = 1 + \frac{3}{7} = \frac{10}{7}$$

$$b = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3}}} = 0 + \cfrac{1}{1 + \cfrac{3}{7}} = 0 + \frac{7}{10} = \frac{7}{10}$$

And we can see that a and b are reciprocals of each other.
More generally, if

$$b = 0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \ddots}} ,$$

We can see that

$$\frac{1}{b} = a_0 + \cfrac{1}{a_1 + \ddots} ,$$

Which is equal to a.

8. How would you express the number -1.17 as a continued fraction? Is there more than one way?

Solution: The first method is to compute the continued fraction for 1.17 and then negate the expression.

$$1.17 = [1; 5, 1, 7, 2]$$

And so,

$$-1.17 = -[1; 5, 1, 7, 2]$$

The second method is a little more complicated. Our motivation here is to think of an equivalent statement that allows us to keep the fractional part of the mixed number positive. Notice that,

$$-1.17 = -2 + 0.83$$

Then we can calculate the continued fraction of 0.83 and get

$$0.83 = [0; 1, 4, 1, 7, 2]$$

Putting it together we have

$$-1.17 = -2 + 0.83 = [-2; 1, 4, 1, 7, 2].$$