Grade 6 Math Circles

November 3rd, 2021

Linear Relations

Variables and Relations

A **variable** is a placeholder for an unknown numerical value in an equation. Usually, a variable is represented by a letter in the English alphabet, like x, or Greek alphabet, like ϕ. A **coefficient** is a numerical or constant quantity that multiplies a variable. Normally, we forgo writing the multiplication symbol and place the coefficient right in front of a variable (i.e. $4 \times x = 4x$).

When we have multiplication between two numbers a and b, instead of writing $a \times b$ we often write it as either $a(b)$ or $(a)(b)$, which are all equivalent (i.e. $2 \times 3 = 2(3) = (2)(3)$).

In many cases, there are a set number of values for a variable.

Example 1

For the equation $2x + 1 = 3$, there is only one possible value of x, which is $x = 1$. We can show that this is true by

$$\begin{aligned} 2x + 1 &= 3 \\ 2x + 1 - 1 &= 3 - 1 && \text{(subtract 1 from both sides)} \\ 2x &= 2 \\ \frac{2x}{2} &= \frac{2}{2} && \text{(divide both sides by 2)} \\ x &= 1 \end{aligned}$$

In other cases, when there are multiple variables, there are an infinite number of values for the variables. These are known as **relations** since there is a relationship between the variables.

Example 2

For the equation $2x + 1 = y$, there are an infinite number of possible values for x and y. The table below gives a few of these possible values:

x	0	1	2	3	4	5	...
$y = 2x + 1$	1	3	5	7	9	11	...

In Example 2, note that each value of y is calculated by plugging each value of x into the equation $y = 2x + 1$. For example, when $x = 2$, then $y = 2x + 1 = 2(2) + 1 = 4 + 1 = 5$. These corresponding values of x and y are called **ordered pairs**, and can be written as (x, y). So, from Example 2, we have the ordered pairs $(0, 1)$, $(1, 3)$, $(2, 5)$, $(3, 7)$, $(4, 9)$, $(5, 11)$, and infinitely many more. For the purposes of this lesson, we will only be dealing with relations between 2 variables, but do note that it is possible to have any number of variables.

Graphing Relations

The relationship between two variables can be graphed on a Cartesian Coordinate Plane using ordered pairs. In order to this, we must follow the steps below:

1. Rewrite the equation so that one of the variables (usually y) is isolated on one side of '='.
2. Using a table, or another method, write down a few of the ordered pairs from the relation.
3. Plot the ordered pairs as points on the Cartesian Coordinate Plane, where the x-values are on the horizontal axis (x-axis) and the y-values are on the vertical axis (y-axis).
4. Connect the points by drawing a line that passes through each them and extends past them.

Example 3

Graph the equation $3x + y = 4$ using the steps above.

Solution 3

1. Our first step is to isolate y on one side of the equal sign. This is shown below

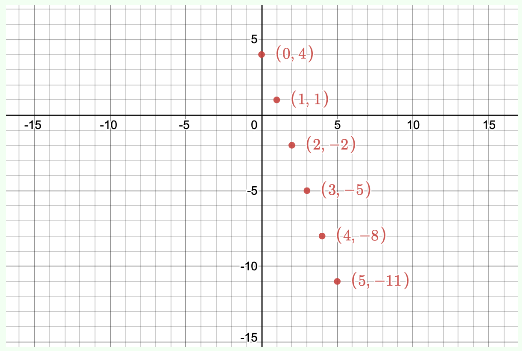
$$\begin{aligned} 3x + y &= 4 \\ 3x + y - 3x &= 4 - 3x \\ y &= -3x + 4 \end{aligned}$$

2. We then use the equation $y = -3x + 4$ to get the following table of values:

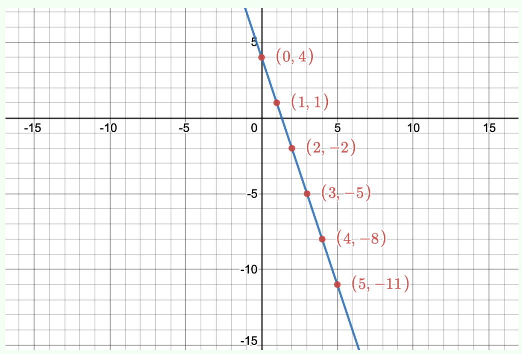
x	0	1	2	3	4	5	...
$y = -3x + 4$	4	1	-2	-5	-8	-11	...

This gives us the ordered pairs $(0, 4)$, $(1, 1)$, $(2, -2)$, $(3, -5)$, $(4, -8)$, $(5, -11)$.

3. Plotting these points on a Cartesian Coordinate Plane gives us:



4. Finally, we draw a line that passes through each point and extends past them.



Activity 1

The link here will take you to the Desmos Graphing Calculator. Take some time to play around with it and understand how it works. Once you feel comfortable with it, use it to graph the following equations.

(a) $x = -7$

(b) $y = 4$

(c) $2x - 3y = 5$

(d) $-x = y - 1$

Linear Relations

A **linear relation**, or **linear equation**, is a relationship between two variables, usually x and y, where the graph of the relationship is a straight line. In most cases, x is called the **independent variable**, and y is called the **dependent variable**. This means that the value of y depends on the value of x. The general form of a linear relation is:

$$y = mx + b$$

This is known as the **slope-intercept form**. Here, m is called the **slope**, and b is called the **y-intercept**. For example, the equation $y = 2x + 1$ is a linear relation, with $m = 2$ and $b = 1$. Both m and b are constant numerical values, not variables.

Activity 2

Determine if the following equations are linear relations or not.

(a) $y = x$

(b) $y = 0$

(c) $7x - 4y = 19$

(d) $y = yx + x$

The y-intercept, b, is simply the value of y when $x = 0$, which is where the graph of the relation intersects the y-axis. The slope, m, describes the both the *direction* and the *steepness* of the line.

- If $m > 0$, then the line is **increasing** (goes up from left to right).
- If $m < 0$, then the line is **decreasing** (goes down from left to right).
- If $m = 0$, then the line is horizontal, also known as a **constant** relationship.
- If m is *undefined* (division by 0), then the line is vertical.

Lines with m-values closer to 0 are less steep than lines with m-values further from 0.

We can calculate m by looking at the graph of the linear relation and picking any two points of the line. Then, we calculate the difference between the y-values (called the **rise**), and the difference between the x-values (called the **run**). Finally, we divide the rise by the run to get m. This formula is given below:

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are points on the line.

Example 4

What is the slope of the line that contains the points $(7, -1)$ and $(-2, 35)$?

Solution 4

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{35 - (-1)}{(-2) - 7} = \frac{35 + 1}{-2 - 7} = \frac{36}{-9} = -4$$

Activity 3

For the following points on a line, calculate m and determine if the line is increasing, decreasing, horizontal, or vertical.

(a) $(9, -2)$ and $(9, 4)$

(b) $(1, -1)$ and $(4, -10)$

(c) $(-9, -11)$ and $(0, 7)$

(d) $(0, -1)$ and $(100, -1)$

Note that it doesn't matter which of pairs is (x_1, y_1) and (x_2, y_2), the answer will be the same.

We can also calculate m and b using the equation of a linear relation $y = mx + b$.

Example 5

Suppose the graph of a linear relation contains the point $(-7, 12)$.

(a) Determine b if $m = -3$.

(b) Determine m if $b = 17$.

Solution 5

(a) We substitute the values $m = -3$, $x = -7$ and $y = 12$ into $y = mx + b$ to get:

$$\begin{aligned} mx + b &= y \\ (-3)(-7) + b &= 12 \\ 21 + b &= 12 \\ b &= 12 - 21 \\ b &= -9 \end{aligned}$$

(b) We substitute the values $b = 17$, $x = -7$ and $y = 12$ into $y = mx + b$ to get:

$$\begin{aligned} mx + b &= y \\ m(-7) + 17 &= 12 \\ -7m &= 12 - 17 \\ -7m &= -5 \\ m &= \frac{5}{7} \end{aligned}$$

Applications of Linear Relations

Like many things, linear relations have practical applications and can be used to represent or solve real-world problems, where the components of linear relations will represent aspects of these problems.

- The variables x and y will represent the two aspects of the problem that we wish to measure, with y representing the aspect that depends on the aspect represented by x.
- The slope m will represent any sort of rate or change that occurs as x increases or decreases.
- The y-intercept b will represent any sort of constant or initial value that doesn't change as x increases or decreases, and is the value of y when $x = 0$.

  For example, if a streaming service has an initial fee of \$9 and a monthly fee of \$5, then x represents the number of months, y represents the cost of the streaming service, the slope represents the monthly fee ($m = 5$) and the y-intercept represents the initial fee ($b = 9$).

Example 6

Suppose the cost of a membership for a gym is an initial startup fee of \$25, and then an additional \$6 per month.

(a) How can we represent this a linear relation?

(b) What is the cost of a 10-month membership?

(c) After how many months does a membership cost \$199?

Solution 6

(a) First let us define variables to represent unknown values. We have that the cost of the membership depends on the number of months someone is a member. So we can say that the number of months is the independent variable x, and the cost of the membership is the dependent variable y.

Next, we have that the cost increases by \$6 for each month a person is a member, which means that if x increases by 1, then y increases by 6, so $m = 6$. Additionally, we have to include the initial startup fee of \$25, which is constant for any number of months, so $b = 25$. Thus, we get the following equation to represent the cost of the membership for any number of months:

$$y = 6x + 25$$

(b) Next, to find the cost of a 10-month membership, we simply substitute $x = 10$ into the equation and solve for y, which gives:

$$\begin{aligned} y &= 6x + 25 \\ &= 6(10) + 25 \\ &= 60 + 25 \\ &= 85 \end{aligned}$$

Thus, a 10-month membership costs \$85.

(c) Finally, to find the number of months for a membership costing \$199, we substitute $y = 199$ into the equation and solve for x, which gives:

$$\begin{aligned} 6x + 25 &= y \\ 6x + 25 &= 199 \\ 6x &= 199 - 25 \\ 6x &= 174 \\ x &= \frac{174}{6} \\ x &= 29 \end{aligned}$$

Thus, a membership costs \$199 after 29 months.

Note that in a case like this we would usually put restrictions on the values of x and y for practical reasons. Specifically, we would say that $y \geq 0$ since we can't have a negative cost, $x \geq 0$ since we can't have a negative number of months, and if $x = 0$ then $y = 0$, because it doesn't make sense that a 0-month membership would cost \$25.

Activity 4

The cost of a certain electrician is as follows: an initial flat fee of \$200, and then an hourly fee of \$45.

(a) Represent this as a linear relation. Be sure to state what each component represents.

(b) What is the cost if the electrician works for 6 hours?

(c) How long would the electrician have to work for the cost to be \$605?