Grade 7/8 Math Circles

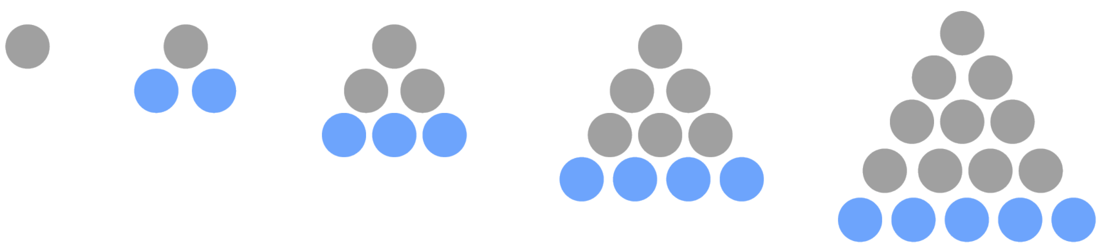
November 3, 2021

Polygonal Numbers

Triangular Numbers

Suppose you are given a big pile of coins, and are asked to arrange the coins into an equilateral triangle. What are some possible coin arrangements?

Starting from one coin, you can add rows of coins to expand the triangle like so:



When we look at the number of coins used to make each arrangement, we end up with a sequence of **triangular numbers**:

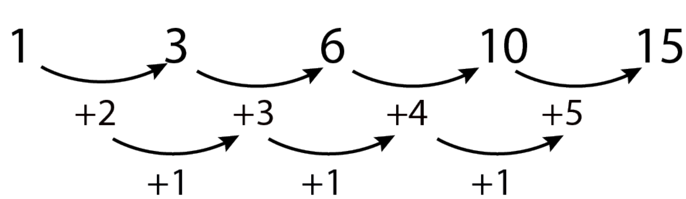
$$T = \{1, 3, 6, 10, 15, ...\}$$

We will use the uppercase letter T to denote triangular numbers, and T_n to denote the n^{th} triangular number. For example, from the images above, we can see that $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, and so on.

Notice that for triangle n, we simply add a new row of coins with length n to the previous triangle. Therefore, the number of coins needed to make the n^{th} triangle is

$$T_n = 1 + 2 + 3 + ... + n$$

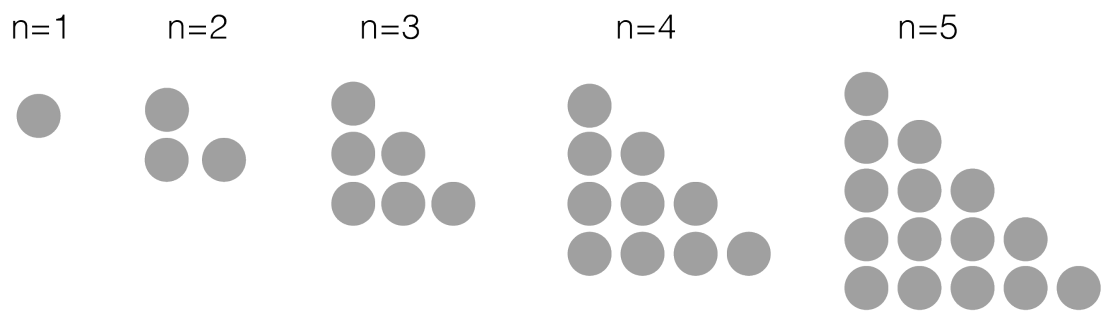
Let us compute the first and second difference between the terms in this sequence.



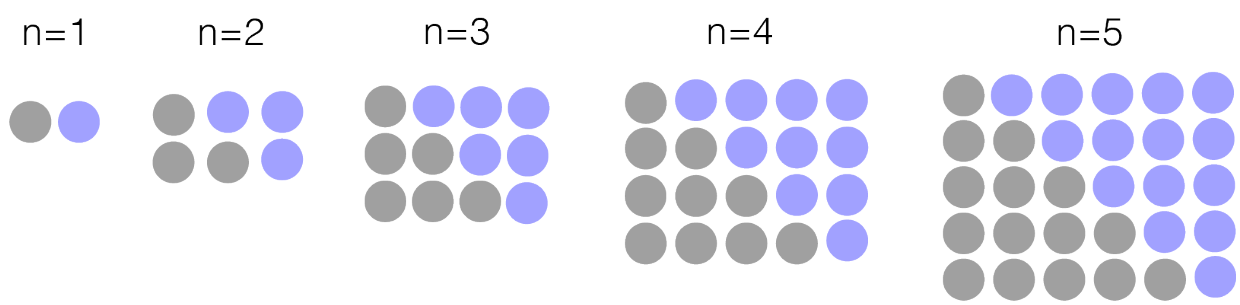
Since this sequence has a common second difference, we know that it's a **quadratic sequence**.

We already know how to find the closed-form formula for this sequence from the previous lesson. Instead, we will find the closed-form formula in a visual way in this lesson.

Let us arrange the coins into right-angled triangles for this demonstration.



If we double the number of coins, we can make a rectangle with dimensions n by $(n+1)$



The formula for the area of a rectangle is $width \times length = n \times (n+1)$, and we can also use this to calculate the number of coins in each rectangle. Knowing that the triangular numbers are half of each rectangle, we can deduce that the closed-form formula for the triangular numbers is

$$T_n = \frac{1}{2}n(n+1) = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

Exercise 1

Use systems of equation (method used in the previous lesson) to find the closed-form formula for the triangular numbers to double-check the formula we just derived.

Triangular numbers appear in the real world as well, such as ten-pin bowling and fifteen-ball pool. Triangular numbers can also be used to solve interesting math problems. Let's take a look at one of the most famous counting problems - The Handshake Problem.

The Handshake Problem

The problem is as follows.

Suppose there is a party with n invited guests, and each guest would like to shake hands with every other guest to introduce themselves. How many handshakes must occur?

Let us start from the most simple situation and work our way up to see if we can find a pattern.

- $n = 1$: if there is a single guest, 0 handshakes occur, since there is no point in shaking hands with yourself.
- $n = 2$: if there are 2 guests, then 1 handshake occurs.
- $n = 3$: if there are 3 guests (let's call them A, B, and C), A shakes hands with both B and C (2 handshakes), and then B and C shake hands (1 handshake). In total, $2 + 1 = 3$ handshakes occur.
- $n = 4$: if there are 4 guests (A, B, C, D), A shakes hands with B, C, and D (3 handshakes), then B shakes hands with C and D (2 handshakes), and finally C shake hands with D (1 handshake). In total, $3 + 2 + 1 = 6$ handshakes occur.

You may have noticed a pattern. Each time another guest enters, they have to shake hands with everyone who is already present (this is similar to "adding a new row of coins" in each subsequent triangle). The rest of the guests will shake hands with each other in exactly the same way as they did in the previous scenario with one less guest (just like how the rest of the triangle does not change when we add a new row).

To generalize, in a party with n guests, the n^{th} person shakes hands with the $(n - 1)$ people already present at the party, and the number of handshakes that occurred between those $(n - 1)$ people is $(n - 2) + (n - 3) + (n - 4) + \cdots + 2 + 1$. Therefore the number of handshakes would be

$$(n - 1) + (n - 2) + (n - 3) + ... + 1 = T_{n-1}$$

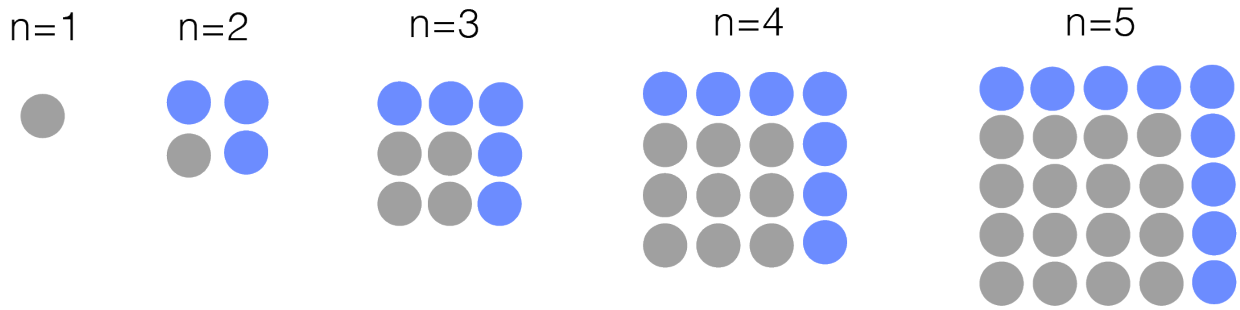
which is the $(n - 1)^{\text{th}}$ triangular number.

Exercise 2

How many handshakes will occur in a party of 35 guests?

Square Numbers

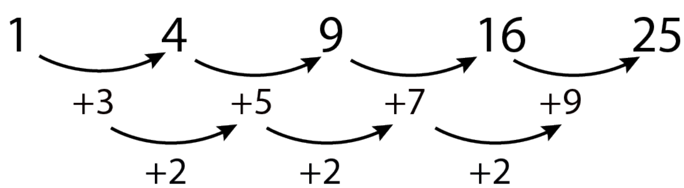
Square numbers are constructed by arranging identical objects in rows and columns to make a square.



Once again, we can construct a sequence using the number of objects in each square.

$$S = \{1, 4, 9, 16, 25, ...\}$$

Computing the first and second difference we have



Since the second difference is constant, this is another quadratic sequence. The closed-form formula for this sequence is rather obvious, since the area of a square is $(\textit{side length})^2$, so the formula for finding the n^{th} term of the sequence of square numbers is:

$$S_n = n^2$$

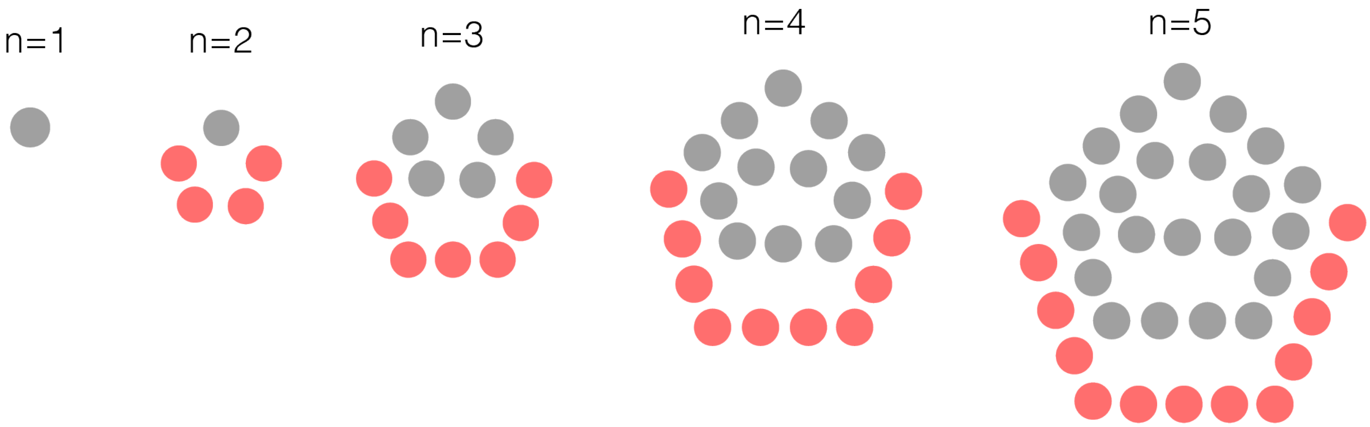
Notice that each time we construct a new square, we are adding a certain number of objects (shown in blue) to the previous square. Starting at $n = 1$, we add 3 objects to get from S_1 to S_2, add 5 objects to get from S_2 to S_3, add 7 objects to get from S_3 to S_4, and so on. The number of objects being added each time n increases are actually consecutive odd integers

$$1 + 3 + 5 + 7 + 9 + ...$$

Therefore we can also conclude that the n^{th} square number ($S_n = n^2$) is also equal to the sum of the first n consecutive odd positive integers.

Pentagonal Numbers

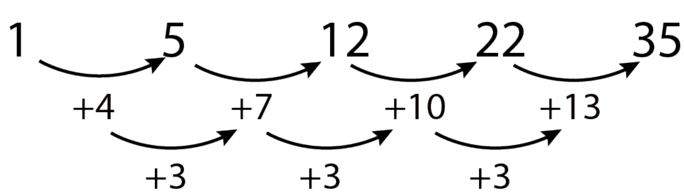
Pentagonal numbers are constructed by arranging identical objects to make a pentagon. Each new pentagon has a side length that is 1 unit longer than the previous pentagon, and contains the previous pentagon inside of it.



Counting the number of objects in each figure, we obtain the sequence of pentagonal numbers:

$$P = \{1, 5, 12, 22, 35, ...\}$$

Computing the first and second difference we have



Here, the second difference is constant once again. Therefore, this is also a quadratic sequence. The closed-form formula for this sequence is given by

$$P_n = \frac{3}{2}n^2 - \frac{1}{2}n = \frac{3n^2 - n}{2}$$

You can also derive this formula yourself using systems of equations, as we did in Exercise 1 for the triangular numbers.

You may have noticed something about the second differences in the sequences we have talked about so far. The second common difference is 1 for the triangular numbers, 2 for the square numbers, and 3 for the pentagonal numbers. This is not a coincidence. In fact, every time the polygon's number of sides increases by 1, the common second difference of its sequence also increases by 1.

Recall that in the previous lesson, the common second difference corresponds to $\frac{1}{2}a$ in the closed-form formula of a quadratic sequence, so there may be a pattern in the closed-form formula for different sequences of polygonal numbers.

Below are the first 8 sequences of polygonal numbers and their closed-form formula (rewritten for pattern recognition).

Number of Sides	**Sequence**	**Closed-Form Formula for the n^{th} Term in the Sequence**
3	Triangular	$\frac{1}{2}(n^2 + n)$
4	Square	$\frac{1}{2}(2n^2 - 0n)$
5	Pentagonal	$\frac{1}{2}(3n^2 - n)$
6	Hexagonal	$\frac{1}{2}(4n^2 - 2n)$
7	Heptagonal	$\frac{1}{2}(5n^2 - 3n)$
8	Octagonal	$\frac{1}{2}(6n^2 - 4n)$
9	Nonagonal	$\frac{1}{2}(7n^2 - 5n)$
10	Decagonal	$\frac{1}{2}(8n^2 - 6n)$

Notice that every formula starts of with the fraction $\frac{1}{2}$. The coefficient of n^2 within the bracket increases by 1 each time, while the coefficient of n within the bracket decreases by 1 each time. If we let s represent the number of sides of a regular polygon and relate s to the coefficients of n^2 and n, we can create a generalized formula for the n^{th} term of the sequence of s-sided polygonal numbers.

Notice that the coefficient of n^2 within the bracket is equal to $(s-2)$, and the coefficient of n within the bracket is equal to $-(s-4)$ (you may verify this with a calculator or by mental math). Therefore, the **generalized closed-form formula** for the n^{th} term of the sequence of s-sided polygonal numbers is

$$\frac{1}{2}((s-2)n^2-(s-4)n)=\frac{(s-2)n^2-(s-4)n}{2}$$

Exercise 3

What is the 12^{th} term of the sequence of heptagonal numbers?