Grade 7/8 Math Circles

October 6th, 2021

Probability - Solutions

Lesson

Activity 1

Use the three formulas listed above to answer the following questions.

(a) What is the probability of rolling an even number on a standard 6-sided die ('die' is singular for 'dice')?

(b) What is the probability of drawing a face-card (Jack, Queen, King) from a standard deck of 52 cards?

(c) How many times does the event A occur if out of 25 total outcomes, A has a probability of 0.64?

(d) How many total outcomes are there if the event B occurs 54 times and has a probability of 0.4?

Activity 1 Solution

(a) The probability of rolling an even number is $\frac{3}{6} = \frac{1}{2} = 0.5$

(b) The probability of drawing a face-card is $\frac{12}{52} = \frac{3}{13}$

(c) Since $S = 25$ and $P(A) = 0.64$, we have $|A| = S \times P(A) = 25 \times 0.64 = 16$

(d) Since $|B| = 54$ and $P(B) = 0.4$, we have $S = \frac{|B|}{P(B)} = \frac{54}{0.4} = 135$

Activity 2

Suppose we have a standard deck of 52 cards and we randomly draw a card. Determine the probability of the following intersection of events. Show your work.

(a) $A \cap B$ = the card is a **Spade** AND the card is a **7**.

(b) $A \cap B$ = the card is a **Spade** AND the card is a **Diamond**.

(c) $A \cap B$ = the card is greater than **5** AND the card is not a face-card.

Activity 2 Solution

(a) Of the 52 cards in the deck, there is only 1 card that is both a Spade and a 7 (the 7 of Spades), so $P(A \cap B) = \frac{1}{52}$

(b) Of the 52 cards in the deck, there are 0 cards that are both a Spade and a Diamond, so $P(A \cap B) = \frac{0}{52} = 0$

(c) Of the 52 cards in the deck, there are 20 cards that are greater than 5 and not face-cards (6, 7, 8, 9, 10 of each suit), so $P(A \cap B) = \frac{20}{52} = \frac{5}{13}$

Activity 3

Determine if the following events are independent or dependent. If they are independent then solve for the probability of the intersection, $A \cap B$.

(a) When flipping a coin, A = first flip is Heads, and B = second flip is Tails.

(b) When flipping a coin, A = first flip is Heads, and B = second flip is Heads.

(c) When rolling a 6-sided die, A = value is even, and B = value is 2.

(d) In a student council election, there are 10 candidates, with 4 of them in grade 7 and 6 of them in grade 8. There are two positions available: President and Treasurer; with A = the President is in grade 7, and B = the Treasurer is in grade 8.

Activity 3 Solution

(a) The outcomes of different coin flips do not affect one another, so the events A and B are independent. Thus, $P(A \cap B) = P(A) \times P(B) = 0.5 \times 0.5 = 0.25$.

(b) The outcomes of different coin flips do not affect one another, so the events A and B are independent. Thus, $P(A \cap B) = P(A) \times P(B) = 0.5 \times 0.5 = 0.25$

(c) Since the events are for the same dice-roll, instead of different dice-rolls, the event A will certainly affect B, and vice versa, so the events are dependent.

(d) The outcome of either position will result in 1 less person being eligible for the other position, which will affect the probability. Thus, the events A and B are dependent.

Activity 4

Find $P(A \mid B)$ for each of the following using the formulas on the previous page.

(a) $P(A \cap B) = 0.4$ and $P(B) = 0.6$

(b) $P(A \cap B) = 0$ and $P(B) = 0.7$

(c) $P(A \cap B) = 0.5$ and $P(B) = 1$

Activity 4 Solution

(a) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.4}{0.6} = 0.24$

(b) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0}{0.7} = 0$

(c) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.5}{1} = 0.5$

Problem Set

1. A 6-sided die is rolled, where each side has a number between 1 and 6.

   (a) What is the probability of rolling a 2?

   (b) What is the probability of rolling a 6?

   (c) What is the probability of rolling a 1 or 5?

   (d) What is the probability of rolling a number greater than or equal to 3?

   Solution:

   (a) The probability of rolling a 2 is $\frac{1}{6}$

   (b) The probability of rolling a 6 is $\frac{1}{6}$

   (c) The probability of rolling a 1 or 5 is $\frac{2}{6}$ or $\frac{1}{3}$

   (d) The probability of rolling a number greater than or equal to 3 is $\frac{4}{6}$ or $\frac{2}{3}$

2. There are 50 monkeys in an exhibit at the zoo. Of these 50 monkeys, 24 are male and the remaining are females. Suppose we were to randomly pick a monkey, with the event $A =$ the monkey is a male.

   (a) Determine $P(A)$.

   (b) What is $\overline{A}$ and $|\overline{A}|$?

   (c) Use any method to determine $P(\overline{A})$.

   Solution:

   (a) $P(A) = \frac{24}{50} = 0.48$

   (b) $\overline{A} =$ the monkey is a female, and $|\overline{A}| = 26$

   (c) $P(\overline{A}) = \frac{26}{50} = 0.52$ or $P(\overline{A}) = 1 - P(A) = 1 - 0.48 = 0.52$

3. There are two types of birds in a bird sanctuary: Blue Jays and Cardinals. Let the event A = the bird is a Cardinal, with $|A| = 42$ and $P(A) = 0.6$.
   (a) What is $\overline{A}$ and $P(\overline{A})$?
   (b) Determine S.
   (c) Use any method to determine $|\overline{A}|$.

> *Solution*:
> (a) $\overline{A}$ = the bird is a Blue Jay, and $P(\overline{A}) = 1 - P(A) = 1 - 0.6 = 0.4$
> (b) Rewriting the formula $P(A) = \dfrac{|A|}{S}$ gives $S = \dfrac{|A|}{P(A)} = \dfrac{42}{0.6} = 70$
> (c) $|\overline{A}| = S - \overline{A} = 70 - 42 = 28$, or $|\overline{A}| = S \times P(\overline{A}) = 70 \times 0.4 = 28$

4. Suppose we have two independent events A and B, where $P(A) = 0.35$ and $P(B) = 0.8$. Determine $P(A \cap B)$.

> *Solution*: Since A and B are independent, $P(A \cap B) = P(A) \times P(B) = 0.35 \times 0.8 = 0.28$

5. Suppose we have two events A and B, where $P(A \cap B) = 0.3$ and $P(B) = 0.75$. Determine $P(A \mid B)$.

> *Solution*: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.3}{0.75} = 0.4$

6. For two events A and B, we have $P(A) = 0.5$, $P(B) = 0.6$, and $P(A \cap B) = 0.3$.
   (a) Determine $P(A \mid B)$.
   (b) Determine $P(B \mid A)$.
   (c) Using the results from parts (a) and (b), are A and B independent or dependent?

Solution:

(a) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.3}{0.6} = 0.5$

(b) $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.3}{0.5} = 0.6$

(c) Since $P(A \mid B) = P(A)$, $P(B \mid A) = P(B)$, and $P(A \cap B) = P(A) \times P(B)$, we know that A and B must be independent.

7. In a sack of marbles, the marbles are categorized by the following attributes that have no influence on one another: the marbles are either *transparent* or **opaque**; and the marbles are either red or blue. The events are A = the marble is **opaque**, and B = the marble is red. There are a total of 125 marbles in the sack, with $P(A) = 0.56$ and $P(\overline{B}) = 0.4$.

   (a) Determine S, $|A|$, $|\overline{A}|$, $|B|$, and $|\overline{B}|$.

   (b) Are the events A and B independent or dependent? Explain.

   (c) Determine $P(A \cap B)$, $P(A \cap \overline{B})$, $P(\overline{A} \cap B)$, and $P(\overline{A} \cap \overline{B})$.

   (d) What is the sum of the probabilities from part (c)? Why is this the case?

Solution:

(a) $S = 125$
$|A| = S \times P(A) = 125 \times 0.56 = 70$
$|\overline{A}| = S - |A| = 125 - 70 = 55$
$|\overline{B}| = S \times P(\overline{B}) = 125 \times 0.4 = 50$
$|B| = S - |\overline{B}| = 125 - 50 = 75$

(b) The events A and B are independent because whether a marble is *transparent* or **opaque** does not affect the colour of the marble.

(c) Since the events are all independent:
$P(A \cap B) = P(A) \times P(B) = 0.56 \times [1 - P(\overline{B})] = 0.56 \times [1 - 0.4] = 0.56 \times 0.6 = 0.336$
$P(A \cap \overline{B}) = P(A) \times P(\overline{B}) = 0.56 \times 0.4 = 0.224$
$P(\overline{A} \cap B) = P(\overline{A}) \times P(B) = [1 - P(A)] \times [1 - P(\overline{B})] = [1 - 0.56] \times [1 - 0.4] = 0.44 \times 0.6 = 0.264$
$P(\overline{A} \cap \overline{B}) = P(\overline{A}) \times P(\overline{B}) = [1 - P(A)] \times 0.4 = [1 - 0.56] \times 0.4 = 0.44 \times 0.4 = 0.176$

(d) $P(A \cap B) + P(A \cap \overline{B}) + P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B}) = 0.336 + 0.224 + 0.264 + 0.176 = 1$
The sum of these probabilities is 1 because these are all the possible combinations we can have for the marbles; **opaque** and red, **opaque** and blue, *transparent* and red, or *transparent* and blue.

8. Use the Number of Pairings formula to determine how many pairings there are for the given number of people.

(a) 1 person
(b) 2 people
(c) 10 people
(d) 50 people
(e) 100 people

Solution:

(a) For $n = 1$, Number of Pairings $= \dfrac{n \times (n-1)}{2} = \dfrac{1 \times (1-1)}{2} = \dfrac{1 \times 0}{2} = \dfrac{0}{2} = 0$

(b) For $n = 2$, Number of Pairings $= \dfrac{n \times (n-1)}{2} = \dfrac{2 \times (2-1)}{2} = \dfrac{2 \times 1}{2} = \dfrac{2}{2} = 1$

(c) For $n = 10$, Number of Pairings $= \dfrac{n \times (n-1)}{2} = \dfrac{10 \times (10-1)}{2} = \dfrac{10 \times 9}{2} = \dfrac{90}{2} = 45$

(d) For $n = 50$, Number of Pairings $= \dfrac{n \times (n-1)}{2} = \dfrac{50 \times (50-1)}{2} = \dfrac{50 \times 49}{2} = \dfrac{2450}{2} = 1225$

(e) For $n = 100$, Number of Pairings $= \dfrac{n \times (n-1)}{2} = \dfrac{100 \times (100-1)}{2} = \dfrac{100 \times 99}{2} = \dfrac{9900}{2} = 4950$

9. Use the general formula for the Birthday Problem to determine the probability that at least 2 people have the same birthday for the given number of people. Round to 4 decimal places.

(a) 3 people

(b) 5 people

(c) 8 people

(d) 10 people

Solution:

(a) For $n = 3$, $P(A) = 1 - \left[\dfrac{365}{365} \times \dfrac{364}{365} \times \dfrac{363}{365}\right] = 1 - 0.9918 = 0.0082$

(b) For $n = 5$, $P(A) = 1 - \left[\dfrac{365}{365} \times \dfrac{364}{365} \times \dfrac{363}{365} \times \dfrac{362}{365} \times \dfrac{361}{365}\right] = 1 - 0.9729 = 0.0271$

(c) For $n = 8$, $P(A) = 1 - \left[\dfrac{365}{365} \times \dfrac{364}{365} \times \dfrac{363}{365} \times \dfrac{362}{365} \times \dfrac{361}{365} \times \dfrac{360}{365} \times \dfrac{359}{365} \times \dfrac{358}{365}\right] = 1 - 0.9257 = 0.0743$

(d) For $n = 10$, $P(A) = 1 - \left[\dfrac{365}{365} \times \dfrac{364}{365} \times \dfrac{363}{365} \times \dfrac{362}{365} \times \dfrac{361}{365} \times \dfrac{360}{365} \times \dfrac{359}{365} \times \dfrac{358}{365} \times \dfrac{357}{365} \times \dfrac{356}{365}\right] = 1 - 0.8831 = 0.1169$

Bonus Questions

10. A new backpack is released that comes in 1 of 50 different colours. Every student in a school orders exactly one of these backpacks online. Unfortunately, they are not given a choice of colour and the colour of the backpack is picked at random, where each of the 50 colours have the same likelihood of being picked. Suppose that for a given number of students, we want to find the probability for the event A = at least 2 people have the same coloured backpack.

    (a) Using a similar method to the Birthday Problem, find a general formula for $P(A)$ where n is the number of people.

    (b) Determine $P(A)$ for $n = 10$. Round to 4 decimal places.

    (c) For what value of n is $P(A) = 0.45$ approximately?

Solution:

(a) For n = the number of people, $P(A) = 1 - \left[\frac{50}{50} \times \frac{49}{50} \times ... \times \frac{50-n+1}{50}\right]$

(b) For $n = 10$, $P(A) = 1 - \left[\frac{50}{50} \times \frac{49}{50} \times \frac{48}{50} \times \frac{47}{50} \times \frac{46}{50} \times \frac{45}{50} \times \frac{44}{50} \times \frac{43}{50} \times \frac{42}{50} \times \frac{41}{50}\right] =$ $1 - 0.3817 = 0.6183$

(c) For $n = 8$, $P(A) = 1 - \left[\frac{50}{50} \times \frac{49}{50} \times \frac{48}{50} \times \frac{47}{50} \times \frac{46}{50} \times \frac{45}{50} \times \frac{44}{50} \times \frac{43}{50}\right] = 1 - 0.5542 = 0.4458$ which approximates to 0.45.

11. Determine the general formula for the Birthday Problem in the case of a leap year (366 days in a year).

Solution: For n = the number of people, $P(A) = 1 - \left[\frac{366}{366} \times \frac{365}{366} \times ... \times \frac{366-n+1}{366}\right]$