Grade 7/8 Math Circles
October 27, 2021
Linear and Quadratic Sequences - Solutions

Exercise Solutions

Exercise 1

Identify t_2, t_5, and t_8 in the sequence below:

$$\{4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, ...\}$$

Exercise 1 Solution

$t_2 = 7$, $t_5 = 16$, and $t_8 = 25$.

Exercise 2

Identify the linear sequences from the options below. For any sequence that is linear, also state the first differences.

(a) $\{-5,\ 0,\ 5,\ 10\}$

(b) $\left\{\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, ...\right\}$

(c) $\{1,\ 5,\ 1,\ 5,\ 1,\ 5,\ ...\}$

(d) $\{2,\ 4,\ 8,\ 16,\ 32,\ 64\}$

(e) $\{9,\ 9,\ 9,\ 9,\ 9,\ 9,\ ...\}$

Exercise 2 Solution

(a) This is a linear sequence with the common first difference being 5.

(b) This is a linear sequence with the common first difference being -1.

(c) This is not a linear sequence since the differences between terms are alternating between positive 4 and negative 4.

(d) This is not a linear sequence. The difference between the first two terms is positive 2, but the difference between the next two terms is positive 4, and so on.

(e) This is a linear sequence with the common first difference being 0.

Exercise 3

Express $t_n = 3n + 1$ in $t_n = t_1 + a(n-1)$ form.

Exercise 3 Solution

Since $t_1 = 4$, we can express the formula as $t_n = 4 + 3(n-1)$

Exercise 4

What is the 18^{th} term of the sequence with the formula $t_n = -2n + 10$?

Exercise 4 Solution

Substituting $n = 18$ into $t_n = -2n + 10$ we get $t_{18} = -2(18) + 10 = -36 + 10 = -26$, so the 18^{th} term is -26.

Exercise 5

Find the closed-form formula for the sequence $\{-2, 5, 12, 19, 26, ...\}$ in both $t_n = an + b$ and $t_n = t_1 + a(n-1)$ form.

Exercise 5 Solution

We can use $t_2 = 5$ and $t_3 = 12$ to obtain a system of equations.

$$5 = 2a + b \qquad (1)$$

$$12 = 3a + b \qquad (2)$$

Solving the system using substitution we have

$$\begin{aligned} a &= 7 \\ b &= -9 \end{aligned}$$

Therefore, the closed-form formula in both forms are

$$\begin{aligned} t_n &= 7n - 9 \\ t_n &= -2 + 7(n-1) \end{aligned}$$

Exercise 6

What is the 20^{th} term in the sequence above? You may use the formula we derived.

Exercise 6 Solution

Substituting $n = 20$ into the formula $t_n = n^2 + 2n + 3$, we have that

$$\begin{aligned} t_{20} &= (20)^2 + 2(20) + 3 \\ &= 20 \times 20 + 2 \times 20 + 3 \\ &= 400 + 40 + 3 \\ &= 443 \end{aligned}$$

Problem Set Solutions

1. For each sequence below, state if it is linear, quadratic, or neither. For sequences that are linear/quadratic, state the common first/second differences.
   (a) $\{1, 1, 1, 1, 1, ...\}$
   (b) $\{35, 27, 22, 20, 21, 25, ...\}$
   (c) $\{3, 6, 12, 24, 48, ...\}$
   (d) $\{-3, 8, 23, 42, 65\}$
   (e) $\{5, 0, 5, 0, 5, 0, ...\}$

Solution:

(a) Linear with common first difference of 0.

(b) Quadratic with common second difference of 3.

(c) Neither

(d) Quadratic with common second difference of 4.

(e) Neither

2. What is the 6^{th} term of the sequence defined by $t_n = \frac{1}{2}n^2 - 2n + 3$?

Solution:

Substituting $n = 6$ into the formula, we get that

$$\begin{aligned} t_6 &= \frac{1}{2}(6)^2 - 2(6) + 3 \\ &= \frac{1}{2}(36) - 12 + 3 \\ &= 18 - 12 + 3 \\ &= 6 + 3 \\ &= 9 \end{aligned}$$

3. Find the sequence defined by $t_n = \frac{3}{2}n - \frac{1}{2}$, $1 \le n \le 6$. Is this a linear or quadratic sequence?

Solution:

Since the formula provided is in $t_n = an + b$ form, it is a linear sequence. You can also confirm this fact by computing the differences between each term.

To compute the sequence, substitute n into the formula provided to obtain t_n.

$$\begin{aligned} t_1 &= \frac{3}{2}(1) - \frac{1}{2} = \frac{3}{2} - \frac{1}{2} = \frac{2}{2} = 1 \\ t_2 &= \frac{3}{2}(2) - \frac{1}{2} = \frac{6}{2} - \frac{1}{2} = \frac{5}{2} \\ t_3 &= \frac{3}{2}(3) - \frac{1}{2} = \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = 4 \\ t_2 &= \frac{3}{2}(4) - \frac{1}{2} = \frac{12}{2} - \frac{1}{2} = \frac{11}{2} \\ t_2 &= \frac{3}{2}(5) - \frac{1}{2} = \frac{15}{2} - \frac{1}{2} = \frac{14}{2} = 7 \\ t_2 &= \frac{3}{2}(6) - \frac{1}{2} = \frac{18}{2} - \frac{1}{2} = \frac{17}{2} \end{aligned}$$

Therefore, the sequence is $\left\{1, \frac{5}{2}, 4, \frac{11}{2}, 7, \frac{17}{2}\right\}$

4. How many terms are in the sequence $\{3, 10, 17, 24, ..., 101\}$?

Solution:
We can see that this is a linear sequence with 7 being the first common difference. Thus, we can find the closed-form formula for t_n using the first two terms. Substituting the first two values into $t_n = an + b$ we have

$$3 = a + b \qquad (1)$$
$$10 = 2a + b \qquad (2)$$

Rearranging the first equation we have that $b = 3 - a$. Substituting this into the second equation we have

$$\begin{aligned} 10 &= 2a + (3 - a) \\ &= 2a - a + 3 \\ &= a + 3 \end{aligned}$$

Thus, $a = 7$. Substituting the value of a into the first equation we get that $b = -4$. Therefore the closed-form formula for this sequence is $t_n = 7n - 4$. Since 101 is the last term in this sequence, we can substitute $t_n = 101$ into the formula to find its term number, which equals the number of terms in the sequence.

$$101 = 7n - 4$$

Solving this equation gives us $n = 15$. Therefore there are 15 terms in this sequence.

5. For each of the following sequences, compute the closed-form formula for the n^{th} term.

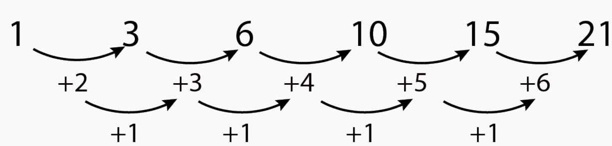
   (a) $\{1, 3, 6, 10, 15, 21, ...\}$

   (b) $\left\{\frac{3}{2}, 4, \frac{13}{2}, 9, \frac{23}{2}, ...\right\}$

   (c) $\{15, 13, 8, 0, -11, ...\}$

Solution:

(a) Computing the first and second differences we have



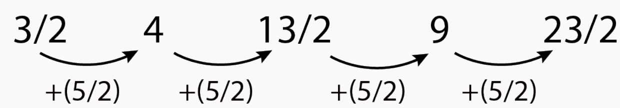
We notice that there is a common second difference, so this is a quadratic sequence. Substituting the first three values into $t_n = an^2 + bn + c$ we have

$$1 = a + b + c \quad (1)$$
$$3 = 4a + 2b + c \quad (2)$$
$$6 = 9a + 3b + c \quad (3)$$

After solving the system we get that $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = 0$.

Therefore the closed-form formula for this sequence is $t_n = \frac{1}{2}n^2 + \frac{1}{2}n$.

(b) Computing the first difference we have

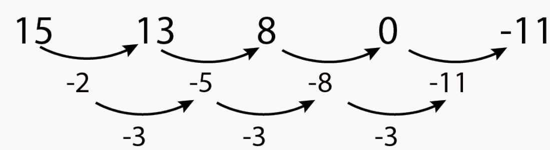


There is a common first difference, so this is a linear sequence.
Since we have the value of a, we can substitute $a = \frac{5}{2}$ into the equation

$$\frac{3}{2} = a + b$$

to obtain $b = -1$. (We obtained the equation above by using $t_1 = \frac{3}{2}$ and $n = 1$.
Therefore the closed-form formula for this sequence is $t_n = \frac{5}{2}n - 1$.

(c) Computing the first and second differences we have



There is a common second difference, so this is a quadratic sequence.
Substituting the first three values into $t_n = an^2 + bn + c$ we have

$$15 = a + b + c \quad (1)$$
$$13 = 4a + 2b + c \quad (2)$$
$$8 = 9a + 3b + c \quad (3)$$

After solving the system we get that $a = -\frac{13}{2}$, $b = \frac{5}{2}$, and $c = 14$.
Therefore the closed-form formula for this sequence is $t_n = -\frac{3}{2}n^2 + \frac{5}{2}n + 14$.

6. $3x + 1$, $5x - 3$, and $6x - 1$ are consecutive terms in a linear sequence. Find the value of x.

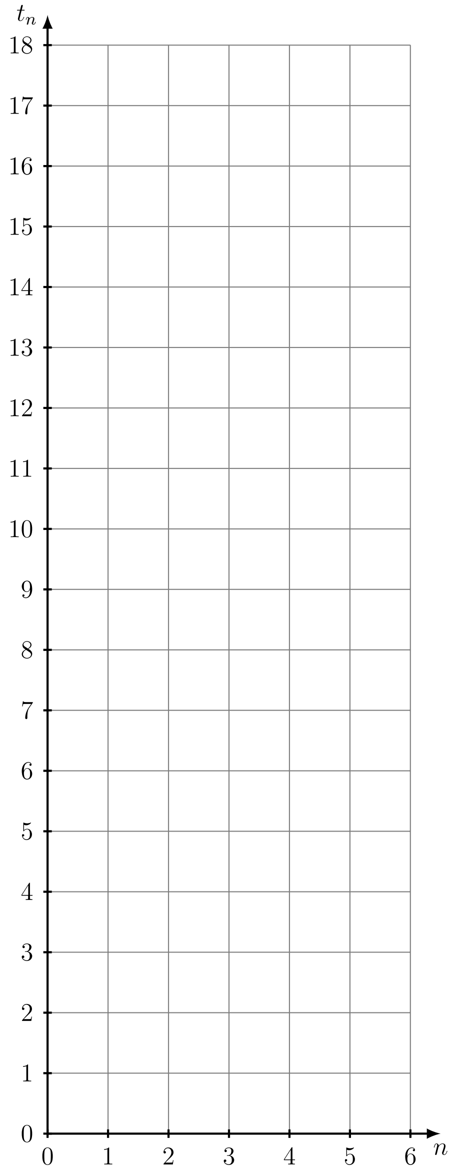
Solution: A linear sequence has a constant first difference, which means the differences between each consecutive term are equal. Therefore, the differences between $3x+1$, $5x-3$, and $6x - 1$ are also equal. Thus, we can solve for x by equating the differences between each consecutive term. (If this is unfamiliar to you, feel free to look at online resources about simplifying and solving linear equations)

$$(5x - 3) - (3x + 1) = (6x - 1) - (5x - 3)$$
$$5x - 3 - 3x - 1 = 6x - 1 - 5x + 3$$
$$2x - 4 = x + 2$$
$$x = 6$$

7. In the grids provided below, plot the following sequences using n as the x-axis and t_n as the y-axis, or $(n,\ t_n)$. For example, if the first term of the first sequence is 3, plot a point on the coordinate (1, 3). Then, connect the points using a line. What do you notice?

   (a) The linear sequence $\{1,\ 5,\ 9,\ 13,\ 17\}$

   (b) The quadratic sequence $\{1,\ 2,\ 4,\ 7,\ 11,\ 16\}$


t_n
18
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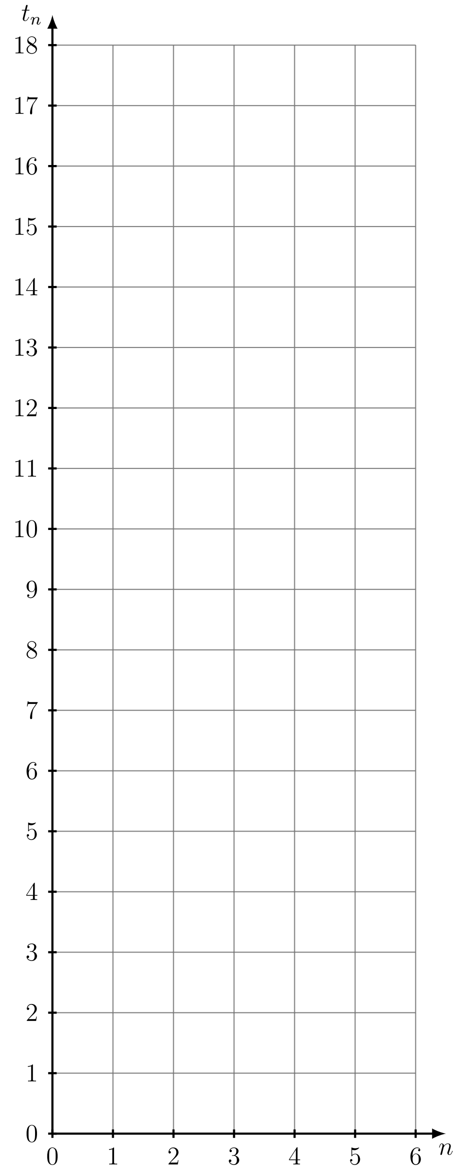
0 1 2 3 4 5 6
n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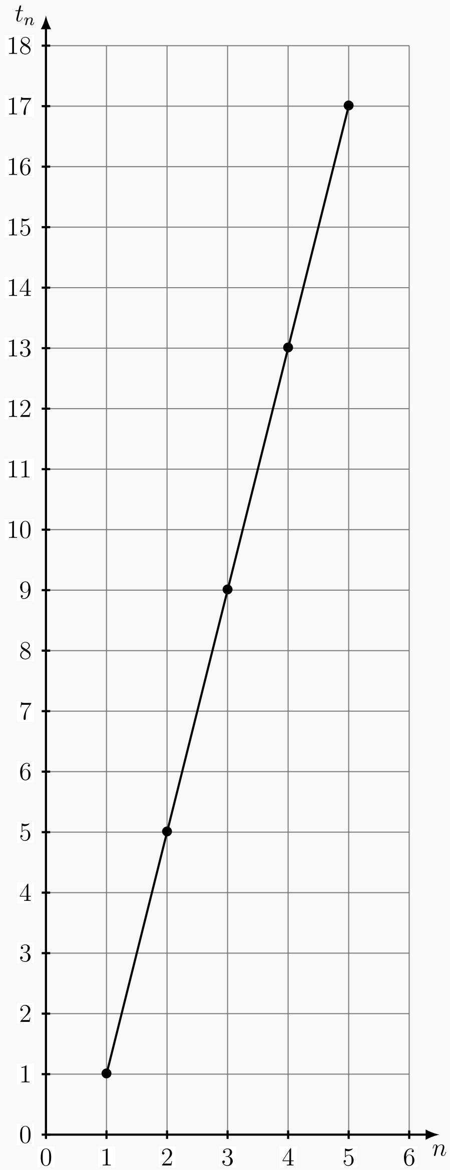
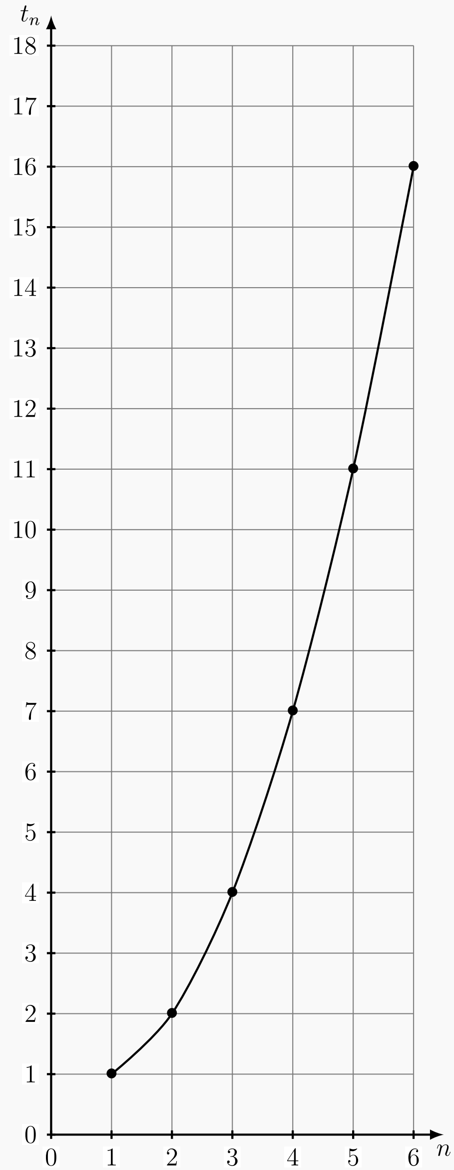
Solution:





We notice that the linear sequence makes a straight line, which does not surprise us, since the root word of "linear" is "line". Also notice that the quadratic sequence makes a curve instead of a straight line. This is also a property of quadratic functions, which you will discover in your future math courses!

8. Here are some sequences that are not linear nor quadratic. Find the next 3 terms in each sequence by finding patterns.

   (a) $\{1, 3, 9, 27, 81, 243, ...\}$

(b) $\{4, 5, 9, 14, 23, 37, ...\}$

(c) $\{1, 8, 27, 64, 125, 216, ...\}$

(d) $\{1, 7, 21, 46, 85, 141, ...\}$

Solution:

(a) Each term is the previous term multiplied by 3. Therefore the next three terms are

- $t_7 = 243 \times 3 = 729$
- $t_8 = 729 \times 3 = 2187$
- $t_9 = 2187 * 3 = 6561$

Sequences where each term is found by multiplying the previous term by a fixed constant are called **geometric sequences**.

(b) After the first two terms, each term is the sum of the previous two terms. Therefore the next three terms are

- $t_7 = 23 + 37 = 60$
- $t_8 = 37 + 60 = 97$
- $t_9 = 60 + 97 = 157$

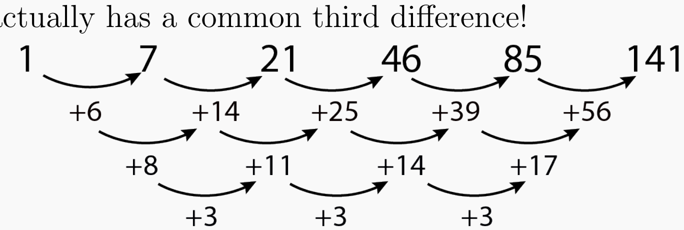
You may have heard of the **Fibonacci Sequence**, which shares a similar pattern.

(c) Each term is the term number multiplied by itself three times, or (n^3). Therefore the next three terms are

- $t_7 = 7^3 = 343$
- $t_8 = 8^3 = 512$
- $t_9 = 9^3 = 729$

Solution:

(d) This sequence actually has a common third difference!



Therefore the next three terms are

- $t_7 = 141 + (56 + 17 + 3) = 217$
- $t_8 = 217 + (76 + 20 + 3) = 316$
- $t_9 = 316 + (99 + 23 + 3) = 441$

Do you notice a pattern of how we get to the subsequent term?