Grade 11/12 Math Circles - Fall 2021
Circles, Ellipses, and Astrophysics
Part 1: Fundamentals

Over the course of two lessons, we will take a look at out-of-this-world geometry (pun intended). In this first lesson, we will build our foundational understanding of circles and ellipses. We will also introduce some astrophysical terminology and key results, and an application of circles and ellipses in the context of satellites. In the second lesson, we will step it up a notch, with some famous (very non-trivial) proofs of the key results introduced in this first lesson. We will then play with real data to extend our knowledge and come to terms with one of the field-changing results in astrophysics: the concept of dark matter.

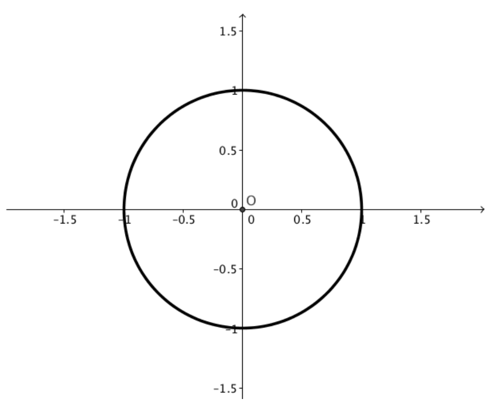
Circles

A **circle**, graphically speaking, is a set of points that is **equidistant** from a central point. This distance is known as the circle's **radius**. It is important to note that the radius is a **constant** value for a circle.

Exercise 1

For this exercise, you will need a piece of string, a piece of tape (or thumbtack), a piece of paper, and a pencil/pen. Using these items, and the definition of a circle above, describe how you would draw a circle, and draw one.

The **unit circle**, depicted below, is the circle centred at the origin with radius 1.



The standard form of the equation of a circle is:

$$(x-a)^2 + (y-b)^2 = r^2$$

where

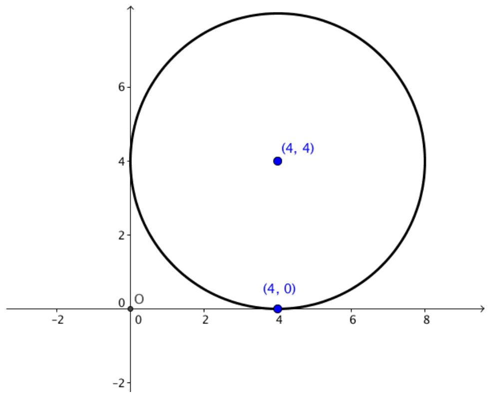
$$\begin{aligned} a &= \text{displacement rightwards from the origin} \\ b &= \text{displacement upwards from the origin} \\ r &= \text{radius of the circle} \end{aligned}$$

Exercise 2

(a) What is the equation of the unit circle?

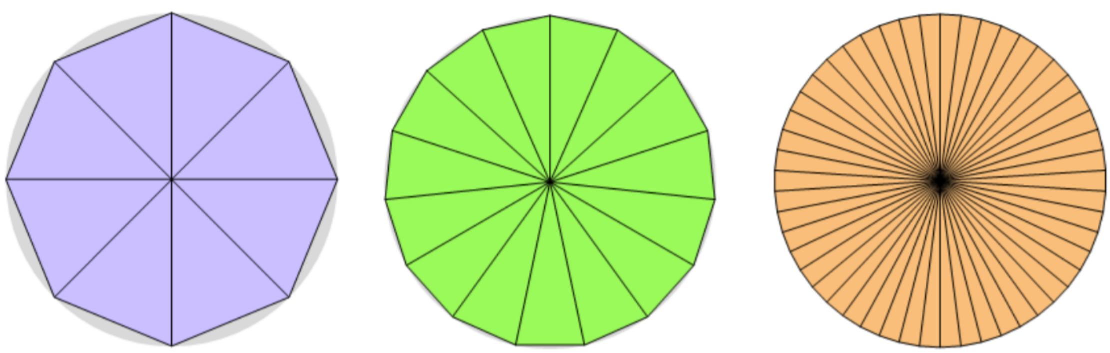
(b) Draw the circle with equation $(x+2)^2 + (y-3)^2 = 4$

(c) What is the equation of the circle pictured below? (The centre is $(4, 4)$.)



An interesting side-adventure in the world of circles, which we will see crop up again in some form in the second lesson, is the idea of approximating a circle as a regular (all sides equal and interior angles equal) n-sided polygon.

Consider constructing several n-sided polygons made up of n isosceles triangle slices, whose two equal sides are of length 1. Below, you will see such a construction for a regular octagon, 15-gon, and 50-gon. It is visually clear that as we increase n for such a construction, we approach a unit circle.



Pictures and values generated by demonstration found at http://demonstrations.wolfram.com/ApproximatingPiWithInscribedPolygons/

Exercise 3

(a) Explain why the angle between the two equal sides of length 1 for each isosceles triangle is $\dfrac{360°}{n}$.

(b) Show that the area of each isosceles triangle is $\frac{1}{2}\sin(\frac{360°}{n})$. You may find the trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$ useful.

(c) Explain why the area of the n-gon is $\frac{n}{2}\sin(\frac{360°}{n})$.

(d) Use this formula to find the area of each n-gon above, as well as a regular 3000-gon, to six decimal places. What are the areas approaching? Why is this expected?

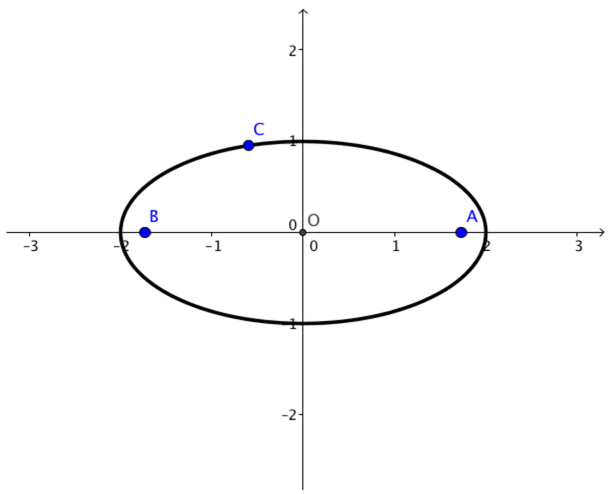
Ellipses

An **ellipse**, graphically speaking, is a set of points such that the sum of the distance from each of two specified fixed points is constant. Each fixed point is called a **focus**, or together they are known as the **foci**. Instead of a radius, an ellipse has a **minor axis** (the shorter one) and a **major axis**. The foci are equidistant from the centre of the ellipse and lie on the major axis.

Exercise 4

For this exercise, you will need a piece of string, two pieces of tape (or thumbtacks), a piece of paper, and a pencil/pen. Using these items, and the definition of an ellipse above, describe how you would draw an ellipse, and draw one. If you get stuck, search the internet! You'll need this technique soon.

An example of an ellipse with major axis 4 and minor axis 2, centred at the origin and with axes corresponding with the graph looks like this:



Remember, by definition, wherever the point C lies on the ellipse, we should find that $BC + AC$ remains constant.

For the sake of ease, we will always centre our ellipses at the origin.

The standard form of the equation for an origin-centred ellipse with major axis corresponding to the x-axis and minor axis corresponding to the y-axis, with foci at $(\pm f, 0)$ is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

The standard form of the equation for an origin-centred ellipse with major axis corresponding to the y-axis and minor axis corresponding to the x-axis, with foci at $(0, \pm f)$ is:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \tag{2}$$

Where

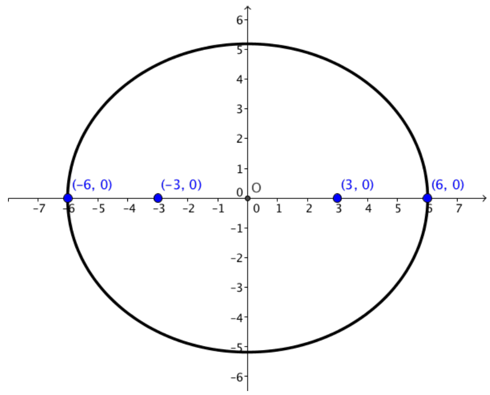
$$\begin{aligned} a &> b \\ a^2 &= f^2 + b^2 \\ a &= \text{half the length of the major axis} \\ b &= \text{half the length of the minor axis} \end{aligned}$$

Exercise 5

(a) What is the equation of the ellipse pictured on the previous page? What are the coordinates of the foci?

(b) Draw the ellipse with equation $\dfrac{x^2}{36} + \dfrac{y^2}{81} = 1$. What are the coordinates of the foci?

(c) What is the equation of the ellipse pictured below? (The foci are $(\pm 3, 0)$.)



We will now work towards showing the initial definition of an ellipse (which you used to create a string technique) holds for equation (1) of an ellipse.

Exercise 6

(a) Draw a general ellipse of the form (1). Then, by using the position where your pen/pencil is at the leftmost point of the ellipse in the string technique, show that the string's length must be $2a$.

(b) Now, using the position where your pen/pencil is at the topmost point of the ellipse in the string technique, show that $f = \sqrt{a^2 - b^2}$ geometrically. (Note: I gave you this fact earlier, but now you are showing it!)

(c) Now consider an arbitrary position for your pen/pencil on the ellipse at some point (x, y) as shown below. Show that $l_1^2 = (\frac{x}{a}\sqrt{a^2 - b^2} + a)^2$ and $l_2^2 = (\frac{x}{a}\sqrt{a^2 - b^2} - a)^2$.

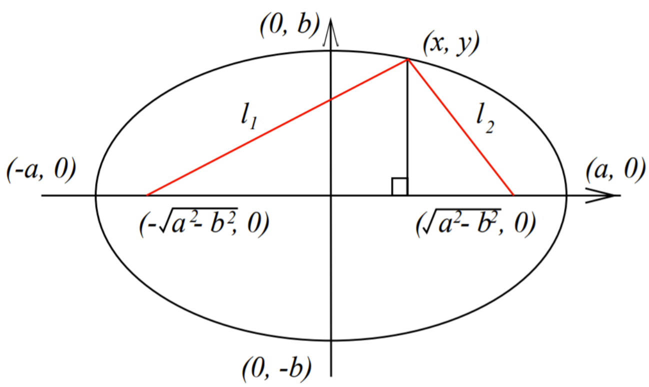
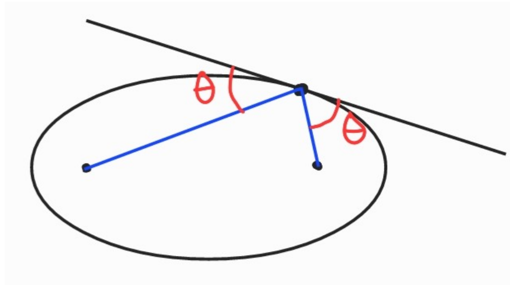


Image credit: Hall & Higson, 1998

(d) Finally, show that $l_1 + l_2 = 2a$, thus concluding that the string technique matches for any (x, y) on the ellipse as defined by equation (1). (Hint: Ensure you are taking positive square roots, as lengths cannot be negative.)

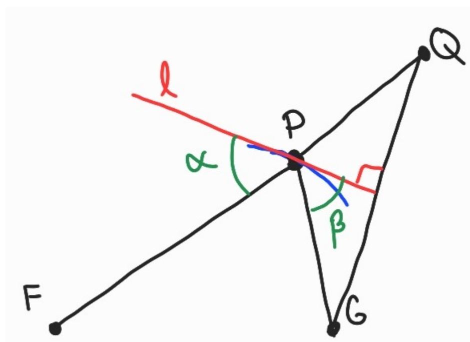
We will now demonstrate two more ways to construct ellipses, the final of which will come in useful during a proof in our second lesson. Firstly, we can define an ellipse as the curve whose tangent (a line which just touches the curve at one point) at any point forms equal angles with the lines to each focus, as depicted below.



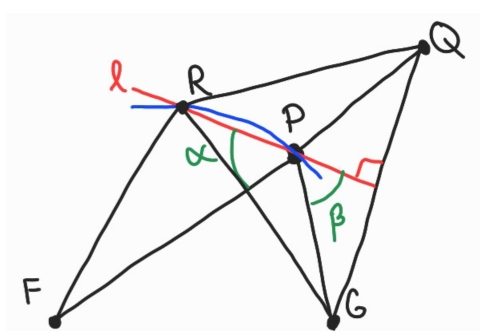
Now, we work to show that this definition matches our initial definition (the string technique).

Exercise 7

(a) Suppose point P is on an ellipse with foci F and G. Draw lines PF and PG. Then, extend extend PF by a length equal to the length of PG, labelling the endpoint Q. Next, draw the perpendicular bisector to QG (the line perpendicular to QG which cuts its length in half), and call it l. Your diagram should look as below. Show that angles α and β are equal, thus making l the line described by the new definition above.



(b) We now demonstrate l must be a tangent line to the ellipse at P. We build a proof by contradiction. First, assume that l intersects the ellipse at another point R, and show that RQ and RG are the same length. The diagram of this setup is below. Then, show that $RF + RG \geq PF + PG$, with equality only holding for when $R = P$ (you may find the triangle inequality useful here - the sum of the lengths of any two sides of a triangle exceeds the length of the third, except in the case where we have a degenerate triangle which is just a line). This shows that l cannot intersect the ellipse at another point R, demonstrating that lines of type l, which are tangent lines, generate an ellipse.



The last way of interest to define an ellipse is the so-called circle construction. In this method, we begin with a circle of centre O, and any fixed non-center point in the circle A. We then pick any point B on the circumference of the circle and connect it to O and A. Then, we find where the perpendicular bisector of AB intersects with OB, and called this point P. We move B around the circumference of the circle, finding all such points P. This set of points forms an ellipse. This is demonstrated in the diagram below.

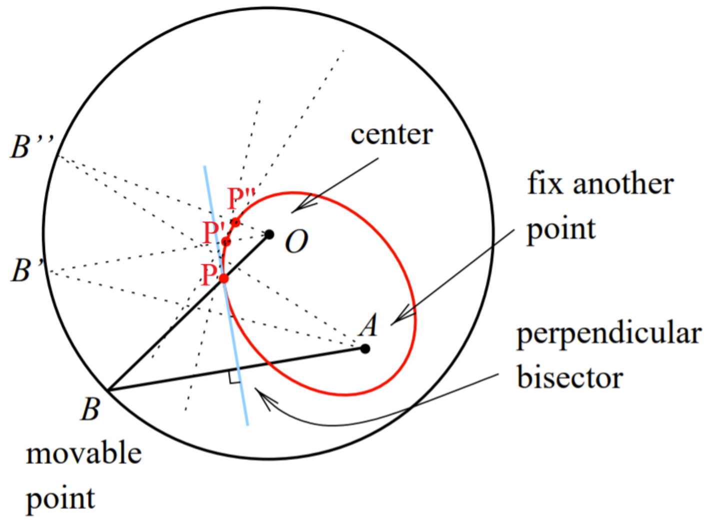


Image credit: Hall & Higson, 1998

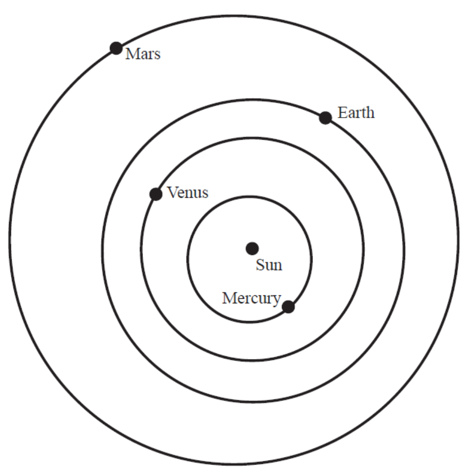
Exercise 8

(a) Using the diagram above, argue that $OP + PA$ is constant (for any location of B around the circumference of the circle), thus satisfying our string technique definition of an ellipse.

(b) Using the diagram above, argue that the perpendicular bisector is a line of type l discussed in Exercise 7(a) and thus satisfies that definition of an ellipse.

Astrophysics

Originally, humans thought that our system revolved around the Earth - a geocentric model often attributed to Ptolemy. Today, we know that the planets orbit around the Sun - a heliocentric model often attributed to Copernicus. What exactly do these orbits look like? Below is a to-scale diagram of the inner planets (planet dots not to scale) that shows the realistic size and shape of their orbits.



A few things should catch your eyes right away:

1. The spacing between orbits is uneven.
2. The orbits look circular.
3. The Sun isn't the exact centre of all the orbits.

What you should realize from a geometric standpoint is that observations 2 and 3 don't work together! We would expect for circular orbits that the Sun should be dead centre, as Copernicus and the early heliocentric astronomers believed. Now, back then, they couldn't plot the orbits as exactly as pictured above so you could forgive them for their misconception.

So what shape are the orbits, and what relationship does the Sun have to the orbits? As you've probably guessed, the answer, as discovered by Kepler, is that these are actually elliptical orbits, with the Sun at one focus!

Eccentricity is a measure of how much an orbit deviates from a circle.

$$\text{eccentricity} = \frac{\text{focus length}}{\text{semimajor axis length}}$$

Where the **semimajor axis** is half of the major axis. A value of 0 represents a perfect circle while the closer the eccentricity gets to 1, the flatter the ellipse appears. The eccentricities of the inner planets are as follows:

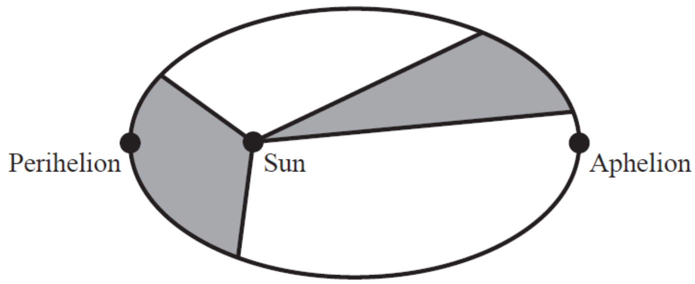
Planet	Eccentricity
Mercury	0.206
Venus	0.0068
Earth	0.0167
Mars	0.0934

We see that we could be forgiven for assuming the orbits were circular. The point where a planet is farthest from the Sun is known as **aphelion** and the point where it is closest to the Sun is known as **perihelion**.

Kepler made two further observations about planetary orbits, and together they make up *Kepler's Three Laws of Planetary Motion*:

1. All planets move in elliptical orbits, with the sun at one focus.
2. The line connecting a planet to the sun sweeps out equal areas in equal times.
3. The square of the period of the orbit of any planet is equal to the cube of the semimajor axis of its orbit times a constant (we say they are proportional to each other). For the planets in our system revolving around the Sun, using units of AU (astronomical units - where 1 AU is the Sun to Earth distance) for distance and years for time, this constant is 1, giving rise to the formula $P^2 = a^3$

Our second lesson's main focus will be to prove these Laws. However, for now, the second law is most approachable and leads us to an interestingly logical conclusion. The image below shows a planet's orbit around the Sun (exaggerated so it's clearly elliptical). The shaded portions have equal area.



Exercise 9

By the second law, these areas must have been traced out in equal time. We can see that the orbital path of the section closer to the Sun is longer than the path farther from the Sun. What does this mean in terms of the planet's motion?

Exercise 10

Jupiter's orbital semimajor axis is 5.203 AU long. Its perihelion distance is 4.950 AU.

(a) What is the eccentricity of Jupiter's orbit?

(b) What is Jupiter's aphelion distance?

(c) What is Jupiter's orbital period in years?

(d) At what distance (in AU) from the Sun is Jupiter moving fastest? Slowest?

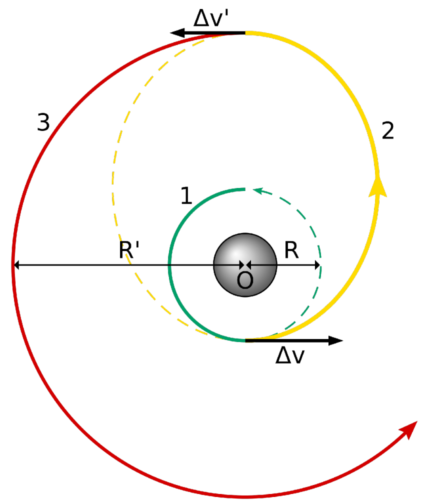
(e) Taking the origin to be the centre of the ellipse, what is the equation for Jupiter's elliptical orbit, taking the major axis to be along the x-axis?

There are upwards of 1500 human-made satellites orbitting the Earth at this very moment! When putting a satellite into orbit, usually adjustments are needed to get it to the required orbital distance. One such technique is known as the Hohmann Transfer.

The Hohmann Transfer is a method that takes an object from one circular orbit to another (either with a larger or smaller radius) around the same central body. This method requires two changes in velocity, usually accomplished through the use of thrusters. A velocity boost (Δv) opposite to the direction of motion will result in a smaller final orbit, while a boost in the same direction will result in a larger final orbit.

The first boost causes the satellite to move onto an elliptical orbit. On this elliptical orbit, the original central body becomes one focus of the ellipse. The spot where the boost was performed will either be the **apogee** or the **perigee** of the elliptical orbit. Apogee means the farthest spot from the focus in question and perigee is the spot nearest that focus. When transferring to a smaller orbit, the boost spot is the apogee, while transferring to a larger orbit makes the boost spot the perigee.

Once the satellite reaches the opposite end of the ellipse, a second boost is performed to bring the satellite into its new circular orbit. The original central point is now the central point of this orbit.



When solving problems about finding the Hohmann Transfer elliptical orbit, you should set your coordinates up so that the origin is at the centre of the ellipse. Given the initial and final circular orbit radii, you will need to find the semimajor axis, focus, and semiminor axis lengths (in that order) to create your equation. It is highly recommended that you draw a diagram for these sorts of problems!

Exercise 11

(a) One Canadian satellite TV provider uses the Nimiq satellites which are in geostationary circular orbits around the Earth. This orbit has a radius of approximately 42300 km (measured from the Earth's centre). Imagine the initial launch successfully puts the satellite in a circular orbit of radius 21000 km (from the centre of the Earth). Find the equation of the Hohmann transfer elliptical orbit, assuming the transfer begins as Nimiq reaches the bottom of its initial circular orbit and the origin is at the centre of the ellipse. Also find the equations of the initial and final circular orbits using the same origin as before.

(b) In the not too distant future, the world's greatest supervillain decides to turn the Earth into a spaceship and drive it to Mars! Assume that the Earth's orbit around the Sun is circular (which it isn't) and that the radius of this orbit is 1 AU. Also assume that Mars also has a circular orbit around the Sun with radius 1.52 AU. Find the equation of the Hohmann transfer elliptical orbit for the journey to Mars, assuming the transfer begins as the Earth reaches the leftmost point of its initial circular orbit and the origin is at the centre of the ellipse. Also find the equations of the initial and final circular orbits using the same origin as before.

While Hohmann transfers are useful when adjusting the orbits of Earth-bound satellites and can also be used to send probes to Mars, this technique is not generally used to send satellites further out in our system.

Instead, gravity assists are used for these missions. Gravity assists have been used on missions to Saturn (Cassini), to a comet (Rosetta), and to interstellar space beyond our system (Voyager 1 in 2012, and Voyager 2 in 2018), to name just a few. Essentially, satellites are swung by planets within our system to redirect and provide a boost in their speed. They pick this speed up from orbital energy, and their momentum keeps them moving onwards rather than getting caught by the planet.