Lesson 1 Exercise Solutions

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1. (a) The game may be described using the table below:

Rose \ Colin	1	2
1	$(-1,1)$	$(1,-1)$
2	$(1,-1)$	$(-1,1)$

Table 1: Game 1(a).

(b) Game 1(b) can also be written using a table as below:

Rose \ Colin	10%	20%	30%	40%	50%
10%	$(10,10)$	$(10,20)$	$(10,30)$	$(10,40)$	$(10,50)$
20%	$(20,10)$	$(20,20)$	$(20,30)$	$(20,40)$	$(20,50)$
30%	$(30,10)$	$(30,20)$	$(30,30)$	$(30,40)$	$(-70,-50)$
40%	$(40,10)$	$(40,20)$	$(40,30)$	$(-60,-60)$	$(-60,-50)$
50%	$(50,10)$	$(50,20)$	$(-50,-70)$	$(-50,-60)$	$(-50,-50)$

Table 2: Game 1(b).

(c) With three players we cannot use a table to describe the game, but we can explain the strategies, and give a full account of how payoffs work for any choice of strategies.

The strategy for each player is a bid. Let r, c, l represent the amounts bid by Rose, Colin, and Larry respectively. Only the bidder who wins the item sees a change in their perceived amount of money and goods, and all other players simply receive payoff 0.

- If $r > c$ and $r > l$, then Rose wins the item of value 20 and pays r, and thus receives payoff $20 - r$.
- Similarly, if $c \geq r$ and $c > l$, then Colin gets payoff $25 - c$.

- Finally, if $l \geq r$ and $l \geq c$ then Larry gets payoff $30 - l$.

2. (a) First, note that every payoff for Colin using B is better than the corresponding payoff using A. Thus, B dominates A, so we may cross out column A from consideration.

   In the reduced game, every payoff for Rose using B is better than for using C, so now for Rose B dominates C, and we may remove row C from consideration.

   In the remaining game of two strategies each, neither player has one strategy that dominates the other. For Rose, her payoff choosing A is better if Colin picks B, but worse if Colin picks C. Conversely, for Colin his payoff for choosing B is better if Rose picks B, but worse if Rose picks A. Thus, this game cannot be simplified further in this manner.

   In general, we can check we are done simplifying by verifying that for every pair of rows r and s, there is at least one column for which r gives better payoff to Rose than s, and one for which s gives better payoff to Rose than r, and similarly for every pair of columns.

   (b) We will go through this one with less words:

   - Row B dominates Row C, so cross out Row C.
   - In what is left, Column C dominates Column A, so cross out Column A.
   - In what is left, Row A dominates Row B, so cross out Row B.
   - In what is left, Column C dominates Column B, so cross out Column B.

   In the end, we are left only with strategy A for Rose and strategy C for Colin, so this must be an equilibrium in pure strategies for this game.

3. (a) There are nine possibilities (R, C) where R is Rose's strategy and C is Colin's strategy.

   - $(1, 1)$: This is not a Nash equilibrium since Rose could increase her payoff from -1 to 1 by playing strategy 2 instead.
   - $(1, 2)$: This is not a Nash equilibrium since Rose could increase her payoff from -1 to 2 by playing strategy 3 instead.
   - $(2, 1)$: Not a Nash equilibrium; Colin could switch to 3.
   - $(2, 2)$: Not a Nash equilibrium, Rose could switch to 3.
   - $(1, 3)$: A Nash equilibrium! Rose can't get a better payoff than -1 in the third column, and Colin can't get a better payoff than 1 in the first row.
   - $(3, 1)$: A Nash equilibrium! Colin can't get a better payoff than -1 in the third row, and Rose can't get a better payoff than 1 in the first column.
   - $(2, 3)$: Not a Nash equilibrium, Rose could switch to 3.
   - $(3, 2)$: Not a Nash equilibrium, Colin could switch to 3.
   - $(3, 3)$: A Nash equilibrium! Neither player can get a better payoff than -1 by switching their own strategy.

   Note the symmetry! The game is symmetric between Rose and Colin, so we'd expect that the Nash equilibria are symmetric as well.

(b) To check that there are Nash equilibria, we just need to confim that neither of them has a reason to change what they're doing. We can see this for the first two equilibria by looking at the table: it's always worse if they both do the same topic, so if one of them has made up their mind, the other should stay out of their way. The third case is more interesting. Here, if Rose picks A, her expected payoff is $-1 \cdot 0.8 + 2 \cdot 0.2 = -0.4$. If she picks B, it's $0 \cdot 0.8 + -2 \cdot 0.2 = -0.4$. So she has no reason to change her strategy; her payoff does not depend on it. The same, symmetrically, is true for Colin.

   So Rose's payoff is 2 in the first case, 0 in the second, and -0.4 in the third. Colin's is 0 in the first case, 2 in the second, and -0.4 in the third. Even though the third Nash equilibrium seems like it's the most fair one, it is actually worse for both Rose and Colin than either of the "unfair" ones.

4. (a) This game is very similar to Rock, Paper, Scissors, and thus we might guess that the equilibrium is similar. In particular, if both players pick 1 and 2 each with probability $\frac{1}{2}$, then it is easy to check that both players have expected payoff 0 no matter whether they change their strategy or not, so this is an equilibrium.

   (b) Any of the four cells of the table immediately outside the bottom-right triangle of negative-payoff cells represents an equilibrium in pure strategies.

   As an example, suppose that Rose always picks 50% and Colin always picks 20%. From Rose's point of view, as long as Colin is only picking 20%, anything other than 50% is strictly worse, so she has no incentive to change strategies. Similarly, if Rose is committed to choosing 50%, then Colin will get a strictly lower payoff from any strategy other than 20%. Thus, this represents an equilibrium, and a similar argument can apply to any one of the other analogous cases.

   (c) Suppose that all three players enter a bid of \$30. Then Larry wins the item, and all players earn payoff 0. All three players' payoffs are unchanged by bidding lower, and all three players' payoffs get strictly worse by bidding higher, so this is an equilibrium.

5. In the original game, if each player bids their true value for the item, then Larry wins the item, but he would also win the item by bidding \$25 and would earn a larger payoff, so this is not an equilibrium.

   However, using the Vickrey auction rules, Larry will pay \$25 regardless of how much he bids at or over \$25, so in particular he has no reason to change from \$30. Thus, using Vickrey auction rules each player bidding their value for the item is an equilibrium.

6. All of these answers can be deduced from the discussion given in the lesson.

   For Rock, Paper, Scissors, we note that if either player is not playing each strategy with probability exactly $\frac{1}{3}$, then some player can improve by switching. Thus, this is an unstable equilibrium.

   For the game of Chicken, consider the equilibrium $(1, 0)$. From our analysis, even if Rose changes r to be slightly smaller than 1, as long as it remains greater than $\frac{997}{999}$ Colin has no incentive to budge from 0, and likewise if Colin changes c to be slightly larger than 0, Rose has no incentive to budge from 1. Therefore, this is a stable equilibrium. It is easy to check that by the same logic $(0, 1)$ is also a stable equilibrium.

Now we consider the equilibrium $(\frac{997}{999}, \frac{997}{999})$. From our analysis, if either player deviates slightly, the other now has the incentive to switch to either 0 or 1. Therefore, this equilibrium is unstable.

7. (a) We may verify that for both players, "take deal" dominates "say nothing", so both players always taking the deal is an equilibrium.

   (b) If both players pick "say nothing", then both end up with a strictly better payoff (-1 vs -6).

   (c) The following table illustrates the payoffs after each game to Player 1, as well as their total across all games:

Player 1 \ Player 2	Jessie	James	Butch	Cassidy	Player 1 Total
Jessie		-1	-10	-10	-21
James	-1		-10	-10	-21
Butch	0	0		-6	-6
Cassidy	0	0	-6		-6

Table 3: Prisoners' Dilemma Payoffs in (c).

   If each pair of players plays a 10-game series, then all of the final payoffs are simply multiplied by 10.

   (d) This time we have the following interactions:

   - When Jessie and James play each other, they both pick "say nothing" in every game, so across 10 games they each have a payoff of -10.
   - When Butch and Cassidy play each other, they both pick "take deal" in every game, so across ten games they each have a payoff of -60.
   - All other matchups proceed in the same way, so we use Jessie and Butch as an example. In the first game, Jessie selects "say nothing" and Butch selects "take deal", so Jessie gets -10 and Butch gets 0. In each of the following nine games, both players pick "take deal", so each get -6. Thus, overall Jessie gets -64 and Butch gets -54.

   Table 4 on the next page illustrates the payoffs after each ten-game series to Player 1, as well as their total across all of the series.

Player 1 \ Player 2	Jessie	James	Butch	Cassidy	Player 1 Total
Jessie		-10	-64	-64	-138
James	-10		-64	-64	-138
Butch	-54	-54		-60	-168
Cassidy	-54	-54	-60		-168

Table 4: Prisoners' Dilemma Payoffs in (d).

8. Suppose both players are playing a mixed strategy. Let the lowest number that Rose selects with positive probability be m (such a number exists since Rose must at least one number with positive probability). Then regardless of Colin's strategy, if Rose instead played $m+1$ instead of m (while keeping the rest of her strategy the same) then she would have strictly better expected payoff. This is true for any mixed strategy, so this game has no equilibrium.

   This does not violate Theorem 2 because the players have an infinite number of strategies, so the conditions of the theorem do not apply.

9. (a) First, suppose that only finitely many climbers choose to ascend. Then there is a last climber l who chooses to ascend. But then each climber after l can change their payoff from 0 to 100 by ascending and thus becoming the last ascending climber (assuming that everyone else doesn't change their strategy), so this cannot be an equilibrium.

   Suppose instead that infinitely many climbers choose to ascend. Then it is impossible to be the last climber to ascend, so all climbers who currently choose to ascend have incentive to instead stay off and increase their payoff from -100 to 0, so this cannot be equilibrium.

   (b) Suppose that $0 < x_i < 1$ for some player i. For each i, let p_i be the probability that at least one climber after i chooses to ascend. Then the expected payoff to player i is

   $$100(1-p_i)x_i - 100p_ix_i = 100x_i(1-2p_i).$$

   If $p_i > \frac{1}{2}$, then $1-2p_i < 0$, so player i maximizes their expected payoff by minimizing x_i, so equilibrium can only occur when $x_i = 0$.

   Likewise, if $p_i < \frac{1}{2}$, then $1-2p_i > 0$, so player i maximizes their expected payoff by maximizing x_i, so equilibrium can only occur when $x_i = 1$.

   In particular, this means that in an equilibrium, for any player i, we can have $0 < x_i < 1$ only when the probability that at least one climber later than them climbs is exactly $\frac{1}{2}$.

   (c) Suppose player i is a player in this equilibrium with $0 < x_i < 1$, so that by the previous part $p_i = \frac{1}{2}$. If $x_j = 0$ for every $j > i$, then clearly $p_i = 0$, which is a contradiction. Hence there is at least one $j > i$ such that $x_j > 0$.

   For this j, if $x_j = 1$, then clearly $p_i = 1$ by definition, again a contradiction, so it follows that $0 < x_j < 1$.

(d) By part (b), since $0 < x_j < 1$, we must have $p_j = \frac{1}{2}$. But note that the probability that at least one player ascends after i includes both the probability that at least one player ascends after j, as well as the probability x_j that j themself ascends, so $p_i \geq x_j + p_j$. In particular, since $x_j > 0$, this means that $p_j < \frac{1}{2}$, a contradiction.

(e) As part (a) shows that no equilibrium in pure strategies is possible, and parts (b)-(d) show that no other equilibrium in mixed strategies is possible, it follows that there are no equilibria for this game.

This does not violate Theorem 2 since this game has infinitely many players, so does not fit the conditions of the theorem.