Grade 7/8 Math Circles

Oct 3/4/5/6

Recursive Sequences - Problem Set

1. For each of the following, identify a pattern and define an infinite sequence which satisfies that pattern.
   (a) 1, 2, 1, 2, 1, ...
   (b) 1, 4, 7, 10, 13, ...
   (c) 1, 11, 111, 1111, ...
   (d) 1, 1, 1, 3, 5, 9, 17, ...
2. Determine whether $\{t_n\}$ is arithmetic, geometric, or neither:
   (a) $t_1 = 1$, t_{n+1} is obtained by adding a 0 at the end of t_n (eg. $t_2 = 10$, $t_3 = 100$, etc.)
   (b) If n is odd, $t_n = n - 1$; if n is even, $t_n = n + 1$
   (c) $t_n = 10000n + 10000$ for $n \geq 1$
   (d) $t_n = n^2$ for $n \geq 1$ (recall: $n^2 = n \times n$)
3. True or False:
   (a) If we remove every odd term of a geometric sequence, the resulting sequence is geometric.
   (b) There is an arithmetic sequence $\{t_n\}$ such that $t_{1000} < t_1 < t_{100} < t_{10}$.
   (c) There is no sequence which is both arithmetic and geometric.
   (d) Challenge: Given 3 numbers, it is always possible to construct an arithmetic sequence which contains all three numbers (i.e. each number is a term in the sequence)?
4. $\{a_n\}$ satisfies $a_1 = 7$ and $a_{n+1} = a_n + 5$ for $n \geq 1$. Find n such that $a_n = 2022$.
5. (*Gauss Gr. 8 2013 #21*): In the grid shown below, the numbers in each row must form an arithmetic sequence and the numbers in each column must form an arithmetic sequence.

5			
			1211
		1013	
12	x		

   What is the value of x?

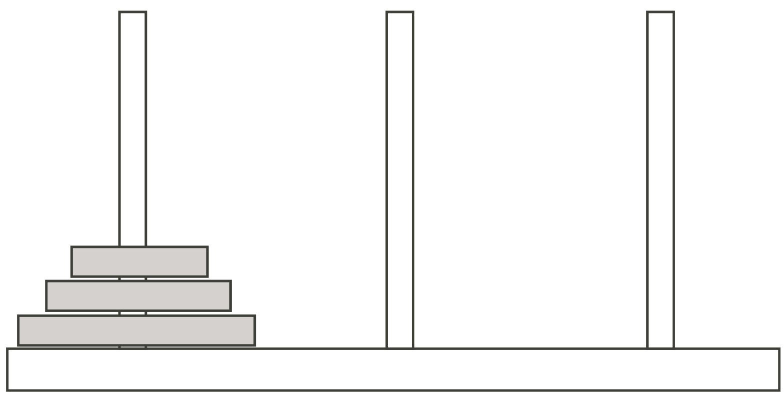
6. Out of the first 2022 Fibonacci numbers, how many are odd?
   Hint: solve the problem with a smaller number than 2022

7. How many ways can one climb a 10-stair staircase by going up 1 or 2 steps at a time?
   Hint: let t_n denote the number of ways one can climb an n-stair staircase by going up 1 or 2 steps at a time. Can you find a recursive formula for the sequence $\{t_n\}$?

8. Alice and Barbara both invest \$100. Alice's money increases by \$10 every year. Barbara's money increases by 5% every year. After 2 years, who will have more money? After 100 years? (a calculator may or may not be needed)

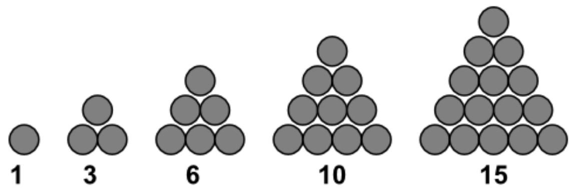
9. Let $\{x_n\}$ satisfy $x_1 = 1$ and $x_{n+2} = x_{n+1} + x_n$ for $n \geq 1$. If x_2 is a positive whole number, find the sum of all values of x_2 for which 45 appears in the sequence.

10. **Tower of Hanoi**: We have three wooden pegs and three rings stacked on the first peg, each ring slightly smaller than the one below it. We want to move the stack of rings to the third peg. However, we may only move one ring at a time, and we may never place a larger ring on top of a smaller one. What is the minimum number of moves required?



    What if we still have three pegs, but we add another disk? What about five disks? Can you find a solution to the problem with three pegs and n disks using recursion?

11. The nth Triangular Number T_n is defined as the number of circles required to make an equilateral triangle (every side is equal) with side length n.



Using the above figure, $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$ and $T_5 = 15$.

a) Find T_6.

b) Find a recursive formula for T_n.

c) Find a closed-form for T_n.
Hint: $1 + 2 + 3 + ... + n = \frac{1}{2}n(n+1)$

12. Consider the arithmetic sequence $\{2n+3\}$ with first term $a = 3$ and common difference $d = 2$.

(a) Let S_n, be the sum of the first n terms of this sequence. For example, $S_3 = 3+5+7 = 15$. Fill out the following table

n	1	2	3	4	5	6	7
S_n			15				

(b) Now consider the sequence $\{S_n\} = S_1, S_2, ..,$. Do you notice a pattern? Can you find a recursive formula? A closed-form?

(c) Challenge: Let $a = a_1$ be the first term of an arithmetic sequence $\{a_n\}$ and let d be its common difference. Find a closed-form formula for the sequence $\{S_n\}$ defined by $S_n = a_1 + a_2 + ... + a_n$.