Grade 7/8 Math Circles

Oct 3/4/5/6

Recursive Sequences - Problem Set Solutions

1. For each of the following, identify a pattern and define an infinite sequence which satisfies that pattern.

   (a) 1, 2, 1, 2, 1, ...

   (b) 1, 4, 7, 10, 13, ...

   (c) 1, 11, 111, 1111, ...

   (d) 1, 1, 1, 3, 5, 9, 17, ...

> *Solution*:
>
> (a) The sequence alternates between 1 and 2. Let $\{a_n\}$ satisfy $a_1 = 1$, $a_2 = 2$ and $a_n = a_{n-2}$ for $n \geq 3$.
>
> (b) The sequence increases by 3 every term. Let $\{t_n\}$ satisfy $t_1 = 1$ and $t_n = t_{n-1} + 3$ for $n \geq 2$.
>
> (c) The n^{th} term is the number with n digits, all of them 1s. Let $\{t_n\}$ satisfy $t_1 = 1$ and $t_n = 10t_{n-1} + 1$ for $n \geq 2$.
>
> (d) Every term is the sum of the three previous terms. Let $\{X_n\}$ satisfy $X_1 = X_2 = X_3 = 1$ and $X_n = X_{n-1} + X_{n-2} + X_{n-3}$ for $n \geq 4$.

2. Determine whether $\{t_n\}$ is arithmetic, geometric, or neither:

   (a) $t_1 = 1$, t_{n+1} is obtained by adding a 0 at the end of t_n (eg. $t_2 = 10$, $t_3 = 100$, etc.)

   (b) If n is odd, $t_n = n - 1$; if n is even, $t_n = n + 1$

   (c) $t_n = 10000n + 10000$ for $n \geq 1$

   (d) $t_n = n^2$ for $n \geq 1$ (recall: $n^2 = n \times n$)

> *Solution*:
>
> (a) geometric: the common ratio is 10 (since adding a 0 to the end of a whole number is the same as multiplying by 10)

(b) neither: $t_1 = 0$, $t_2 = 3$, $t_3 = 2$, $t_4 = 5$. We can check that there is neither a common difference ($t_2 - t_1 = 3 \neq -1 = t_3 - t_2$), nor a common ratio ($\frac{t_3}{t_2} = \frac{1}{3} \neq \frac{5}{2} = \frac{t_4}{t_3}$).

(c) arithmetic: this is a closed-form formula for an arithmetic sequence

(d) neither: $t_1 = 1$, $t_2 = 4$, $t_3 = 9$. We can check that there is neither a common difference ($t_2 - t_1 = 3 \neq 5 = t_3 - t_2$), nor a common ratio ($\frac{t_2}{t_1} = 4 \neq \frac{9}{4} = \frac{t_3}{t_2}$).

3. True or False

(a) If we remove every odd term of a geometric sequence, the resulting sequence is geometric.

(b) There is an arithmetic sequence $\{t_n\}$ such that $t_{1000} < t_1 < t_{100} < t_{10}$.

(c) There is no sequence which is both arithmetic and geometric.

(d) (Challenge): Given 3 numbers, it is always possible to construct an arithmetic sequence which contains all three numbers (i.e. each number is a term in the sequence).

Solution:

(a) True: the ratio between two consecutive terms of the new sequence is $r \times r$, where r is the common ratio of the original sequence. Therefore, the new sequence is geometric.

(b) False: an arithmetic sequence is either increasing (if the common difference is positive), constant (if the common difference is zero), or decreasing (if the common difference is negative). Therefore $t_1 < t_{10}$ implies $t_{10} < t_{100}$ and so $t_1 < t_{100} < t_{10}$ is not possible.

(c) False: consider the sequence of zeros 0, 0, 0, ... which is both arithmetic (common difference is 0) and geometric (common ratio is 0).

(d) False: this is a very tricky question. Recall that π is an irrational number, i.e. it cannot be expressed as $\frac{a}{b}$ for whole numbers a and b. For there to be an arithmetic sequence containing 0, 1 and π, there must be a common difference d such that $0 + n_1 d = 1$ and $0 + n_2 d = \pi$ for whole numbers n_1 and n_2. But then

$$\pi = \frac{\pi}{1} = \frac{n_2 d}{n_1 d} = \frac{n_2}{n_1}$$

and since π is irrational, this is not possible. If we restrict the sequence to be a sequence of whole numbers, then this is actually true.

4. $\{a_n\}$ satisfies $a_1 = 7$ and $a_{n+1} = a_n + 5$ for $n \geq 1$. Find n such that $a_n = 2022$.

Solution: This is an arithmetic sequence with common difference 5 and first term $a_1 = 7$. Using the general closed form for arithmetic sequences,

$$a_n = 5(n-1) + 7 = 5n + 2$$

Solving $a_n = 5n + 2 = 2022$ (so $5n = 2020$ and $n = \frac{2020}{5} = 404$), we obtain $n = 404$.

5. (*Gauss Gr. 8 2013 #21*): In the grid shown below, the numbers in each row must form an arithmetic sequence and the numbers in each column must form an arithmetic sequence.

5			
			1211
		1013	
23	x		

What is the value of x?

Solution: We proceed one row/column at a time:

- Let d_1 be the common difference when going down the first column. Then reading the first column (from top to bottom): $5,\ 5 + d_1,\ 5 + 2d_1,\ 5 + 3d_1 = 23$. Since $5 + 3d_1 = 23$, we have $3d_1 = 18$ and so $d = 6$. We can now fill the first column:

5			
11			1211
17		1013	
23	x		

- We now look at the second row. Let d_2 be its common difference. Then the row is $11,\ 11 + d_2,\ 11 + 2d_2,\ 11 + 3d_2 = 1211$. Solving the last equation, $3d_2 = 1200$ and so $d_2 = 400$. We can then fill up the second row:

5			
11	411	811	1211
17		1013	
23	x		

- We now look at the third row. Let d_3 be its common difference. Then the row is 17, $17 + d_3$, $17 + 2d_3 = 1013$. Solving the last equation, $2d_3 = 996$ and so $d_3 = 498$. We can then fill up the third row:

5			
11	411	811	1211
17	515	1013	1511
23	x		

- Finally, we are able to compute x using the second column. Its common difference is $515 - 411 = 104$. Therefore,

$$x = 515 + 104 = 619$$

6. Out of the first 2022 Fibonacci numbers, how many are odd?
   Hint: solve the problem with a smaller number than 2022

Solution: Let us look at the parity of each term at the start of the Fibonacci sequence:

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55
parity of F_n	odd	odd	even	odd	odd	even	odd	odd	even	odd

Notice there is a pattern: every third term is even, all other terms are odd. We can justify the pattern using:

- the sum of two odd numbers is even
- the sum of an odd number and an even number is odd

Formally, since every third Fibonacci number is even, $2n$ out of the first $3n$ Fibonacci numbers are odd (for any n). Note 2022 is divisible by 3 therefore our final answer is $2022 \times \frac{2}{3} = 1348$.

7. How many ways can one climb a 10-stair staircase by going up 1 or 2 steps at a time?
Hint: let t_n denote the number of ways one can climb an n-stair staircase by going up 1 or 2 steps at a time. Can you find a recursive formula for the sequence $\{t_n\}$?

> *Solution*: Let t_n be the number of ways to climb a n-stair staircase by going up 1 or 2 steps at a time. We can count different possibilities to find $t_1 = 1$ and $t_2 = 2$. For $n > 2$, there are
>
> - t_{n-1} of climbing the n-stair staircase such that we end the climb by going up 1 step (there are t_{n-1} ways of climbing the first $n-1$ steps and we end by going up 1 step.)
> - Similarly, there are t_{n-2} ways of climbing the n-stair staircase such that we end the climb by going up 2 steps.
>
> Therefore, $t_n = t_{n-1} + t_{n-2}$. Then $t_3 = 3$, $t_4 = 5$, $t_5 = 8$, ... and this is just the Fibonacci sequence shifted by one term. We can conclude by computing $t_{10} = F_{11} = 89$.

8. Alice and Barbara both invest \$100. Alice's money increases by \$10 every year. Barbara's money increases by 5% every year. After 2 years, who will have more money? After 100 years? (calculator may or may not be required)

> *Solution*: Let $\{a_n\}$ and $\{b_n\}$ denote Alice's and Barbara's money (in \$) at the end of every year (respectively). Then $a_1 = 110$ and $b_1 = 105$. We can calculate $a_2 = 120$ and $b_2 = 112.5$. So, after 2 years, Alice will have more money. Note $\{a_n\}$ is arithmetic with common difference 10 and $\{b_n\}$ is geometric with common ratio $\%105 = 1.05$. So
>
> $$a_n = 110 + (n-1) \times 10 = 10n + 100$$
> $$b_n = 105 \times 1.05^n$$
>
> After 100 years, Alice will have $a_{100} = 1100$ dollars and Barbara will have $1.05^{100} \times 105$ dollars. Using a calculator, Barbara will have more money after 100 years. We can also reasonably guess this answer since geometric sequences eventually 'grow' faster than arithmetic sequences (see below).
>
> Cool fact: take any arithmetic sequence $\{a_n\}$ with positive first term and positive common difference, and take any geometric sequence $\{b_n\}$ with positive first term and common ratio greater than 1, then there exists a (possibly very large) N such that $a_n < b_n$ for all $n \geq N$.

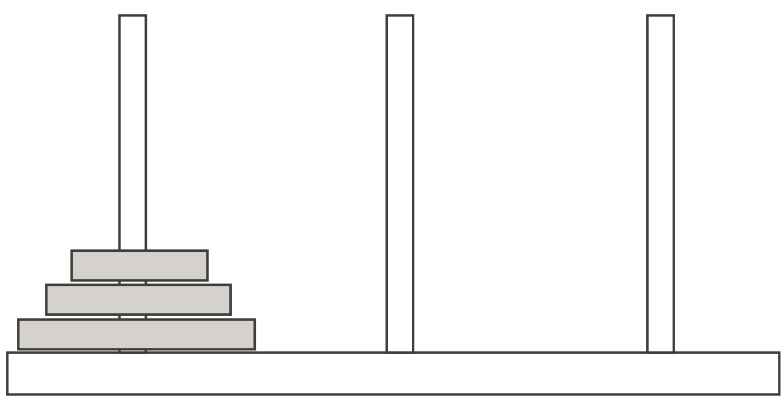
9. Let $\{x_n\}$ satisfy $x_1 = 1$ and $x_{n+2} = x_{n+1} + x_n$ for $n \geq 1$. If x_2 is a positive whole number, find the sum of all values of x_2 for which 45 appears in the sequence?

Solution: Let $x = x_2$. This is the same recursive formula as a Fibonacci sequence. The first terms will be

$$1,\ x,\ 1+x,\ 1+2x,\ 2+3x,\ 3+5x,\ 5+8x,\ 8+13x,\ \text{etc.}$$

If $x \geq 3$ then every term starting from $8 + 13x$ will be greater than 45. We can verify that for $x = 1$ and $x = 2$, this sequence is the Fibonacci sequence so 45 does not appear. We can then proceed to check whether 45 could appear in any of the remaining 7 terms: $5 + 8x = 45$ yields $x = 5$. However, since x is a whole number, none of $1 + 2x$, $2 + 3x$, $3 + 5x$ can be 45 (eg. $3 + 5x = 45$ means $5x = 42$ but 42 is not divisible by 5). We get two final solutions for x using the second term ($x = 45$) and the third term ($x + 1 = 45$ so $x = 44$). Our final answer is $5 + 44 + 45 = 94$.

10. **Tower of Hanoi**: We have three wooden pegs and three rings stacked on the first peg, each ring slightly smaller than the one below it. We want to move the stack of rings to the third peg. However, we may only move one ring at a time, and we may never place a larger ring on top of a smaller one. What is the minimum number of moves required?



What if we still have three pegs, but we add another disk? What about five disks? Can you solve the general problem (3 pegs, n disks) using recursion?

Solution: We provide a general solution to the problem with three pegs, regardless of the number of disks. Let M_n be the least number of moves for the problem if we have n disks. Then $M_1 = 1$ (we can just move the disk directly).

Next, we will try finding a recursive formula. Suppose we have n disks. Note the problem is symmetric: having to stack the disks on the second peg is 'equivalent' to having to stack the disks on the third peg since both are empty. Therefore, M_n is also equal to the number of ways we can stack n disks onto the second peg (assuming the same rules still hold).

At some point, we will need to move the biggest disk to the third peg. To do so, all of the smaller disks need to be stacked on the second peg. The smallest number of moves to stack $n-1$ disks on the second peg is M_{n-1}. We then take 1 move for the big disk. Finally, we need another M_{n-1} moves to move the rest onto the big disk (think about why this is). Hence, we obtain the fomula

$$M_n = 2M_{n-1} + 1$$

so, the sequence $\{M_n\}$ starts with

$$1, 3, 7, 15, 31, ...$$

Note that this is the same pattern as in the Warm-up problem from the lesson. If we add 1 to every term:

$$2, 4, 8, 16, 32, ...$$

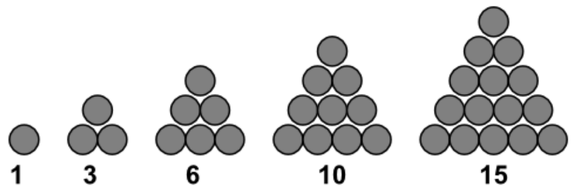
Using this pattern, we can hypothesize that, for n disks and 3 pegs, the answer is

$$M_n = 2^n - 1$$

Finally, we can verify that the recursive formula does indeed hold:

$$M_n = 2^n - 1 = 2 \times 2^{n-1} - 1 = 2 \times (M_{n-1} + 1) - 1 = 2M_{n-1} + 1$$

11. The nth Triangular Number T_n is defined as the number of circles required to make an equilateral triangle (every side is equal) with side length n.



Using the above figure, $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$ and $T_5 = 15$.

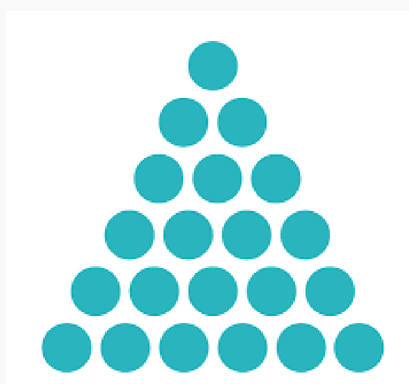
a) Find T_6.

b) Find a recursive formula for T_n.

c) Find a closed-form formula for T_n.
Hint: $1 + 2 + 3 + ... + n = \frac{1}{2}n(n + 1)$

Solution:

(a) $T_6 = 21$:



(b) If we take a triangle of length $n - 1$ and add a 'layer' of n circles to the bottom, we obtain a triangle of length n. Hence,

$$T_n = n + T_{n-1}$$

(c) From the recursive formula,

$$\begin{aligned} T_n &= n + T_{n-1} \\ &= n + (n-1) + T_{n-2} \\ &= n + (n-1) + ... + 2 + T_1 \\ &= n + (n-1) + ... + 2 + 1 \\ &= (n+1) + ((n-1)+2) + ((n-2)+3) + ... + \frac{1}{2}(n+1) \\ &= n(\frac{1}{2}(n+1)) \\ &= \frac{1}{2}n(n+1) \end{aligned}$$

(the last few steps provide an informal proof of the hint)

12. Consider the arithmetic sequence $\{2n+3\}$ with first term $a = 3$ and common difference $d = 2$.

(a) Let S_n, be the sum of the first n terms of this sequence. For example, $S_3 = 3+5+7 = 15$. Fill out the following table

n	1	2	3	4	5	6	7
S_n			15				

(b) Now consider the sequence $\{S_n\} = S_1, S_2, ...,$. Do you notice a pattern? Can you find a recursive formula? A closed-form?

(c) Challenge: Let $a = a_1$ be the first term of an arithmetic sequence $\{a_n\}$ and let d be its common difference. Find a closed-form formula for the sequence $\{S_n\}$ defined by $S_n = a_1 + a_2 + ... + a_n$.

Solution:

(a)

n	1	2	3	4	5	6	7
S_n	3	8	15	24	35	48	59

(b) The difference between terms increases by 2. This comes from the way we constructed the sequence since $S_n = S_{n-1} + 2n + 3$. Then

$$\begin{aligned}
S_n &= S_{n-1} + 2n + 3 \\
&= S_{n-2} + (2(n-1) + 3) + (2n + 3) \\
&= (2(1) + 3) + (2(2) + 3) + ... + (2(n-1) + 3) + (2n + 3) \\
&= 2(1 + 2 + ... + n) + 3n \\
&= 2T_n + 3n \\
&= n(n + 1) + 3n \\
&= n(n + 4)
\end{aligned}$$

(c) The general problem can be solved similarly:

$$\begin{aligned}
S_n &= S_{n-1} + d(n-1) + a \\
&= S_{n-2} + (d(n-2) + a) + (d(n-1) + a) \\
&= (d(0) + a) + (d(1) + a) + ... + (d(n-2) + a) + (d(n-1) + a) \\
&= d(0 + 1 + 2 + ... + (n-1)) + an \\
&= dT_{n-1} + an \\
&= \frac{1}{2}d(n-1)n + an
\end{aligned}$$