Math Circles - Pigeonhole Principle - Fall 2022

Week 1 (Nov 2)

Introduction

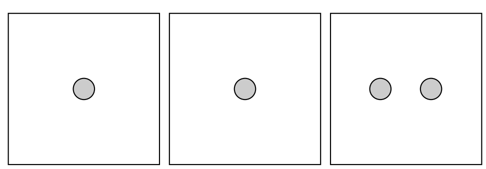
Let's start with a warm up example.

Example. *Suppose you have a drawer filled with red, yellow, and blue socks. If you blindly reach in, how many socks do you need to pull out in order to guarantee that you have two socks of the same colour?*

Solution. We would need to pick out 4 socks. If we only pick out 3 socks, then we could have a red sock, a blue sock, and a yellow sock, each of which is a different colour. So, we need to pick out at least 4 socks. Without loss of generality, assume the first sock we pick out is red.[1] If the second sock is also red, then we have our match, and it doesn't matter which colour the remaining socks are. Otherwise, the second sock is either blue or yellow. Without loss of generality, suppose the second sock is blue. If the third sock is either red or blue, then we have our match, and it doesn't matter which colour the remaining socks are. Otherwise, the third sock must be yellow. Now, since we already have a red sock, a blue sock, and a yellow sock, any choice of colour for the fourth sock will give us a match. So, we are guaranteed a match by choosing 4 socks. ■

This is an application of something called the pigeonhole principle.

Theorem (Pigeonhole Principle)**.** *If you have n holes (where $n > 0$) and m pigeons (where $m > n$), and you put each of the m pigeons into one of the n holes, then one box will end up with at least two pigeons in it.*



In our example about the socks, the pigeons would be the socks, and the holes would be the colours. Since we have 3 colours, then by the pigeonhole principle, we need at least $3 + 1 = 4$ socks to guarantee that we have two socks of the same colour.

Now, let's make things a little more complicated.

Example. *In the first exercise, suppose there are exactly 10 socks of each colour in the drawer. If you blindly reach in, how many socks do you need to pull out in order to guarantee that you have taken out 3 socks of the same colour?*

Solution. We would need to pick out 7 socks. If we only picked out 6 socks, we could have exactly 2 socks of each colour. So, we need to pick out at least 7 socks. Let's look at all possibilities of how we could pick 7 socks from 3 colours:

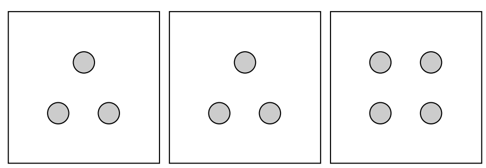
[1] *Without loss of generality* means that even though we are picking a case to analyze, it doesn't matter which case we pick, because the proof is the same for each case.

Red	Blue	Yellow
7	0	0
6	1	0
5	2	0
5	1	1
4	3	0
4	2	1
3	3	1
3	2	2

Notice that, without loss of generality, these are all the cases that we need to consider, because any other case could be obtained by switching the order of the colours at the top of the table. Specifically, there do not exist any other sets of 3 non-negative integers which add up to 7. Since each possibility results in having 3 socks of the same colour, we have shown that we can achieve the desired result by picking out 7 socks. ■

The idea here is similar to our first example: figure out the "worst case scenario" (in our case, this would be when we have equal socks of each colour, as that is how we can pick out the most total socks and have the least of a single colour), and then pick one more sock. It's a more general version of our first example, and is an application of something called the *generalized pigeonhole principle.*

Theorem (Generalized Pigeonhole Principle)**.** *If you have n holes (where $n > 0$) and m pigeons (where $m > n$), and you put each of the m pigeons into one of the n holes, then one box will have at least $\lceil \frac{m}{n} \rceil$ pigeons in it.*[2]



Just like before, in our example about the socks, the pigeons would be the socks, and the holes would be the colours. Since we have 3 colours, and we want to have 3 socks of the same colour, then by the generalized pigeonhole principle, we need need x socks, such that $\lceil \frac{x}{3} \rceil = 3$. The smallest such x is 7, so we need at least 7 socks.

Pigeonhole Proofs

So far, we have seen how to directly apply the pigeonhole principle to solve counting problems, which can be really useful. However, it is used more often in mathematical proofs, often of seemingly-unrelated statements. In these situations, the hardest part is often determining which part of the question relates to the "holes" and which part of the question relates to the "pigeons". The next few examples are meant to demonstrate this.

Example. *Let $x_1, \dots, x_{11} \in \{0, 1, \dots, 99\}$. Show that for some $i, j \in \{1, 2, \dots, 11\}$, it must be the case that $|x_i - x_j| < 10$.*

Solution. Consider the sets $A_0 = \{0, \dots, 9\}$, $A_1 = \{10, \dots, 19\}, \dots, A_9 = \{90, \dots, 99\}$. Notice that, if $a, b \in A_i$, then $|a - b| < 10$. If we let the sets $A_0, \dots, A_9$ be our holes and we let the integers $x_1, \dots x_{11}$ be our pigeons, then by the pigeonhole principle, there must be indices i and j such that x_i and x_j lie in the same set (since we are trying to put 11 integers into 10 sets). So, for these values of i and j, we have that $|x_i - x_j| < 10$. ■

Example. *Suppose a sphere has 5 points drawn on it's surface. Prove that there's a way to cut it in half such that 4 of the points lie on the same closed hemisphere.*[3]

[2] $\lceil x \rceil$ is the *ceiling function* of x, and returns the largest integer greater than or equal to x.

[3] A *closed* hemisphere includes the boundary along which the sphere was cut in half. That is, if there is a point lying on the dividing line between the two hemispheres, you can consider that point to lie on *both* hemispheres.

Solution. Pick two of the points and draw a great circle[4] through them; cut the sphere in half along this great circle to obtain the two hemispheres. Since the hemispheres are closed, then both of these points lie in both hemispheres.

We have three points left to consider. Let these three points be our pigeons, and let the hemispheres be our holes. By the pigeonhole principle, two of them must lie in the same hemisphere. So, that hemisphere contains at least $2 + 2 = 4$ points. ■

Example. *Given two equilateral triangles, show that it is impossible to use them to cover a strictly larger equilateral triangle (without breaking any of the triangles into multiple pieces).*

Solution. In a successful covering of the larger triangle with the smaller triangles, each point of the larger triangle must be covered by one of the smaller triangles. This includes the three vertices of the larger triangle. So, we can let the two smaller triangles be our holes, and the corners of our larger equilateral triangle be our pigeons. By the pigeonhole principle, in any successful covering, it must be the case that two vertices of the big triangle are covered by the same smaller triangle. However, in an equilateral triangle, the maximum possible distance between two points is exactly the distance between two vertices. Since the big triangle is strictly larger than each smaller triangle, and all triangles are equilateral, it is not possible for two vertices of the big triangle to be covered by the same smaller triangle. ■

[4] A *great circle* on a sphere is a circle drawn on the surface of the sphere which has the same diameter as the sphere. In other words, it is the largest possible circle that can be drawn on the sphere. A unique great circle can be drawn through any two points on the sphere, and the shortest distance between two points on the surface of the sphere is the shortest distance along the great circle passing through both of them.