Grade 6 Math Circles
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Introduction to Set Theory

Building Foundations

We will start by defining what a set is.

Definition 1

- A **set** is a collection of objects.
- **Elements** of a set are the objects in the collection.

1. We use the curly brackets, $\{\}$, to denote a set. For example, if we want to express a set containing three elements, a, b, and c, we can write $\{a, b, c\}$, separating each element of the set with a comma.
2. To avoid writing down a set over and over again, we usually use a capital letter to represent the set. We can write $A = \{a, b, c\}$ which means "A represents the set $\{a, b, c\}$."
3. Let $A = \{a, b, c\}$. We write "$b \in A$" to mean that b is an element of the set A. The symbol $\in$ can be read as "element of". Similarly, we write "$x \notin A$" to mean that x is *not* an element of the set A.
4. Two sets that have exactly the same elements are equal. Also, when a set is defined by listing its elements, the order of the elements does not matter. All that matters is *which* objects are in the collection. Note that an element can even appear more than once in the collection/list. Thus all of $\{a, b, c\}, \{c, b, a\}, \{a, a, b, b, c, c\}$ are the same set.
5. Let B represent the set containing all the months in a year. Then we have

$$B = \{\text{January, February, March, April, May, June, July, August, September, October, November, December}\}$$

This is very long and tedious to write.

Alternatively, we can define a set by writing out the pattern(s) that the elements follow. For example, we can write B as:

$$B = \{x \mid x \text{ is a month in a year}\}.$$

This is read "B is equal to the set of all x such that x is a month in a year," and it means that the elements of B are the values of x that make the statement "x is a month in a year" come out true. The statement "x is a month in a year" is called the **elementhood test** (or the **set-builder notation**) for the set.

Exercise 1

Rewrite these set definitions using elementhood tests:

1. $C = \{0, 1, 2, 3, 4, 5, 6, 7, \dots\}$. (note: $\dots$ means the list continues on.)
2. $D = \{\text{a, b, c, d, e, f}, \dots, \text{x, y, z}\}$.
3. $E = \{\text{Clubs, Diamonds, Hearts, Spades}\}$.

Exercise 1 Solution

1. $C = \{x \mid x \text{ is a whole number}\}$.
2. $D = \{y \mid y \text{ is a lower-case letter}\}$.
3. $E = \{z \mid z \text{ is a suit in the standard deck of cards}\}$.

Notice that the definitions of the above sets do not depend on x, y, and z. Thus, you can use any other variable to state the definition of the same set.

We will close this section by asking ourselves an interesting question.

Stop and Think

Can sets be elements of other sets?

Stop and Think Explained

Absolutely yes! An example of such a set is:

$$\{\{a, b\}, \{c\}, \{a, b, c\}\}.$$

The Empty Set

Stop and Think

What if we get really creative with the elementhood test and come up with a test that no element can pass? For example, consider the following set:

$$P = \{x \mid x \text{ is a person taller than 20 feet}\}.$$

Stop and Think Explained

The idea of a set with no elements may sound weird, but this is a natural concept that arises when we think about an elementhood test that no objects can pass.

Definition 2

- The **empty set** (or the **null set**) is the set that has no elements, denoted $\varnothing$ or $\{\,\}$.

We say *the* empty set, not *an* empty set, because there is only one empty set. If we had two different empty sets, since two different sets *must* contain at least one different element, two different empty sets must have different element(s) from each other. However, both sets have NO elements because they are empty! Thus, we cannot have two different empty sets. That is, the empty set is unique. Looking back at our **Stop and Think** above, we can write $P = \varnothing$ because finding a person taller than 20 feet is impossible!

Stop and Think

Can the empty set be an element of a set?

Stop and Think Explained

Absolutely yes! For example, we can have a set containing the empty set:

$$\{\varnothing\}.$$

Keep in mind that this is a totally different set from $\varnothing$. The set $\{\varnothing\}$ has one element, the empty set, but $\varnothing$ has no elements.

Set Operations

Just like how we can add, subtract, multiply, and divide with numbers to get other numbers, we can perform operations on sets by using set operators to get new sets.

Definition 3

Let A and B be sets.

- The **intersection** of A and B, denoted $A \cap B$, contains all the elements that are in both sets. Using the elementhood test, we can write:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

- The **union** of A and B, denoted $A \cup B$, contains the combined elements from both sets. Using the elementhood test, we can write:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

  Note: in mathematics, $x \in A$ or $x \in B$ is always interpreted as $x \in A$ or $x \in B$, or both.

- The **difference** of A and B, denoted $A \setminus B$, contains the elements in A that are not in B. Using the elementhood test, we can write:

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Using the above definitions, we can perform set operations:

Exercise 2

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$. Find the following sets by listing all the elements.

1. $A \cap B$.
2. $A \cup B$.
3. $A \setminus B$.
4. $B \setminus A$.
5. $(A \cup B) \setminus (A \cap B)$.
6. $(A \setminus B) \cup (B \setminus A)$.

Exercise 2 Solution

1. $A \cap B = \{1, 3, 5\}$.
2. $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$.
3. $A \setminus B = \{2, 4\}$.
4. $B \setminus A = \{7, 9\}$.
5. $(A \cup B) \setminus (A \cap B) = \{2, 4, 7, 9\}$.
6. $(A \setminus B) \cup (B \setminus A) = \{2, 4, 7, 9\}$.

A **universal set** is the set of all elements currently under consideration which we can build all other sets from, and it is often denoted with the capital letter, U. A possible universal set for Exercise 2 will be $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Notice that this contains all elements of all related sets, in this case A, B, and as well as elements not in either A or B. When a universal set is defined, we can look at all the elements that are not in a particular set:

Definition 4

Let A be a set and let U be a universal set containing A.

- The **complement** of A, denoted A^C, contains the elements in U that are not in A. Using the elementhood test, we can write:

$$A^C = \{x \mid x \in U \text{ and } x \notin A\}.$$

Notice that $A^C = U \setminus A$.

Exercise 3

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$, as defined in Exercise 2. Let the universal set be $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find the following sets by listing all the elements.

1. A^C.
2. B^C.
3. $A^C \cap B^C$.
4. $(A \cup B)^C$.

Exercise 3 Solution

1. $A^C = \{0, 6, 7, 8, 9\}$.
2. $B^C = \{0, 2, 4, 6, 8\}$.
3. $A^C \cap B^C = \{0, 6, 8\}$.
4. $(A \cup B)^C = \{0, 6, 8\}$.

The idea of taking the complement of a set often comes up in our day-to-day lives. For example, if you are a teacher and you want to contact students who did not show up to the class today, you take the complement of the students who attended the class to find the missing students. In this case, the universal set will be all students in the attendance list.

Subset

Definition 5

Let A and B be sets.

- A is a **subset** of B if every element of A is also an element of B, denoted $A \subseteq B$.
- A and B are said to be **disjoint** if they have no elements in common. In other words, the intersection of A and B is empty, $A \cap B = \varnothing$.

We will now take a look at an example involving subsets.

Example 1

Let $A = \{\text{Math, Science}\}$, $B = \{\text{Math, English, Science, French}\}$, and $C = \{\text{Music, French}\}$.

- The two elements of A, Math and Science, are both also elements of B. Thus, $A \subseteq B$.
- There is no element in common for A and C. Thus, $A \cap C = \varnothing$. In other words, A and C are disjoint.

Is there a set that is a subset of all sets in the entire universe? Is that possible? The answer is yes.

Definition 6

- The empty set, $\varnothing$, is a subset of all sets. We refer to the empty set as the **universal subset**.

Since it contains no elements, *every* element of the empty set belongs to any set. This is an example of a **vacuously true** statement. (There is no case that makes the statement false. Hence, the statement must be true.)

Example 2

Let $A = \{1, 2, 3\}$. The list of all subsets of A is:

$$\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}$$

Stop and Think

Let A and B be sets. What do we know about A and B, if we have $A \subseteq B$ and $B \subseteq A$?

Stop and Think Explained

Then we know that A and B are equal! Recall that two sets that have exactly the same elements are equal. If A and B are not equal, there must be an element in one set that is not in the other. Let's say that there is an element in A which is not in B. But this is against the fact that A is a subset of B. Thus, A and B must be equal.

Types of Numbers

We use all different types of numbers. We can group numbers with the same type in a set.

Definition 7

We will use the following set notations:

- $\mathbb{N}$ will denote the set of all **natural numbers** $\{1, 2, 3, \ldots\}$.
- $\mathbb{W}$ will denote the set of all **whole numbers** $\{0, 1, 2, 3, \ldots\}$.
- $\mathbb{Z}$ will denote the set of all **integers** $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- $\mathbb{Q}$ will denote the set of all **rational numbers** $\left\{\frac{a}{b} \mid a \in \mathbb{Z} \text{ and } b \in \mathbb{N}\right\}$.
- $\mathbb{R}$ will denote the set of all **real numbers**. This is the set which includes "all the numbers" on the (real) number line.

Notice that we have

$$\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Exercise 4

1. Find $\mathbb{W} \setminus \mathbb{N}$ by listing its element(s).
2. Find $\mathbb{Z} \setminus \mathbb{W}$ by using an elementhood test.

Exercise 4 Solution

1. $\mathbb{W} \setminus \mathbb{N} = \{0\}$.
2. This is the set of all negative integers. One possible set definition using an elementhood test is $\mathbb{Z} \setminus \mathbb{W} = \{-n \mid n \in \mathbb{N}\}$.

Now, can you find an element in the set $\mathbb{Q} \setminus \mathbb{Z}$? How about ...

Stop and Think

Find an element in the set $\mathbb{R} \setminus \mathbb{Q}$.

Stop and Think Explained

Some famous irrational numbers include $\sqrt{2} = 1.414213\ldots$, $\pi = 3.141592\ldots$, $\phi = 1.6180339\ldots$ (the Golden Ratio), and $e = 2.71828\ldots$ (Euler's number).

$\mathbb{R} \setminus \mathbb{Q}$ is actually a type of numbers called **irrational numbers**! The decimal form of irrational numbers do not terminate (it just continues on ...) and have no patterns or repeating parts in the mantissa (the "fractional part" or the "part after the decimal point").

Set-theoretic Definition of Whole Numbers

This might be a surprise: we can define whole numbers using only sets! In Zermelo-Fraenkel set theory, the whole numbers are defined *recursively* (using the definition of the previous number) by letting $0 = \varnothing$ and $n + 1 = n \cup \{n\}$. The first few numbers can be defined as:

$$\begin{aligned} 0 &= \{\} &&= \varnothing \\ 1 &= \{0\} &&= \{\varnothing\} \\ 2 &= \{0, 1\} &&= \{\varnothing, \{\varnothing\}\} \\ 3 &= \{0, 1, 2\} &&= \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \end{aligned}$$

Cardinality

How do we know how big a set is?

Definition 8

Let A be a finite set (we will discuss what "finite" means soon).

- We say that A has **cardinality** n, if A contains n elements. We denote this by

$$|A| = n$$

The cardinality of a set is the size of the set.

Exercise 5

Find the cardinality of the following finite sets:

1. $\{1, 3, 5, 7, 9, 11\}$
2. $\{1, 1, 1, 1, 1, 1\}$
3. $\varnothing$

Exercise 5 Solution

1. 6.
2. 1. This is a trick question! $\{1, 1, 1, 1, 1, 1\} = \{1\}$.
3. 0.

Now, let's take a look at what "finite" means.

Definition 9

- A **finite set** is a set that has a finite number of elements. (When you start "counting", you can eventually finish counting the elements in the set).
- A set that is not a finite set is called an **infinite set**.

Stop and Think

What about the cardinality of infinite sets? What is $|\mathbb{N}|$?

Stop and Think Explained

$\mathbb{N}$ is the smallest infinite set and we denote the cardinality of $\mathbb{N}$ as $\aleph_0$. Since we "count" with the natural numbers, we say that $\mathbb{N}$ is countably infinite and we call any sets that have the same cardinality as $\mathbb{N}$ countably infinite as well.

Further Discussion

Stop and Think

Which one is bigger?

$$|\mathbb{Z}| = |\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}|$$

OR

$$|\mathbb{N}| = |\{1, 2, 3, \ldots\}|$$

Stop and Think Explained

Let's pair them up! One way to pair them together is:

- If $n \in \mathbb{N}$ is odd, pair it with $\dfrac{n-1}{2} \in \mathbb{Z}$.
- If $n \in \mathbb{N}$ is even, pair it with $-\dfrac{n}{2} \in \mathbb{Z}$.

$\mathbb{N}$	$\mathbb{Z}$
1	0
2	-1
3	1
4	-2
5	2
6	-3
7	3
8	-4
9	4
10	-5
11	5

Since you can pair every single natural number with every single integer, the cardinality of $\mathbb{N}$ is the same as the cardinality of $\mathbb{Z}$.