Math Circles - Pigeonhole Principle - Fall 2022

Exercises

1. Suppose that S is a set of $n+1$ integers. Prove that S contains distinct integers a and b such that $b-a$ is a multiple of n.

   Solution. For each integer x in S, consider its representation modulo n. There are n possibilities for $x \mod n$, namely, $0, 1, \ldots, n-1$. If we take our possible representations modulo n to be our holes and the integers in S to be our pigeons, then by the pigeonhole principle, there must be two integers a and b in S that are equivalent modulo n. By definition of equivalence modulo n, this means that $b-a$ is divisible by n. ■

2. Let S be a set of 10 distinct integers between 1 and 60, inclusive. Prove that we can choose two disjoint[1] subsets of S (say, S_1 and S_2) such that the sum of the elements in S_1 is equal to the sum of the elements in S_2.

   Solution. Let S be a set of 10 integers between 1 and 60, inclusive. It suffices to show that (non-equal) subsets have the same sum, because after that we can simply remove all common elements to obtain two disjoint subsets with the same sum.

   Since $\emptyset$ contains no elements, the sum of it's elements is 0 (the "empty sum"). Since every other subset contains only strictly positive integers, it is not possible for the elements in any other subset to sum to 0. So, we need not consider $\emptyset$. Also, we need not consider S itself, since every subset of S is completely contained in S will have all of its elements in common with S, so it will not be possible to obtain disjoint subsets in the way described above.

   So, there are $2^{10} - 2 = 1022$ subsets to consider. Since each of the elements are distinct, and at most 60, and each subset can have at most 9 elements, then the largest possible sum we can obtain is $60 + 59 + 58 + 57 + 56 + 55 + 54 + 53 + 52 = 504$. Since each element is at least 1, and each subset has at least one element, then the smallest possible sum we can obtain is 1. So, there are (at most) 504 possible sums. If we take these sums to be our holes, and the 1022 possible subsets to be our pigeons, then by the pigeonhole principle, at least two subsets must have the same sum. ■

3. Show that in any set of 100 integers, one can choose 15 of them such that the difference between any two is divisible by 7.

   Solution. If a and b are integers, then $7 \mid a-b$ if and only if $a \equiv b \mod 7$. There are 7 possible values for an integer modulo 7. If we let these values modulo 7 be our holes and the 100 integers we are given be our pigeons, then by the generalized pigeonhole principle, at least $\lceil \frac{100}{7} \rceil = 15$ of these integers must be equivalent modulo 7. So, we can take 15 of these integers as our desired set. ■

4. Prove that in any set of 100 integers, one can choose a set of at least one number whose sum is divisible by 100.

[1] *Disjoint* means that the sets have no elements in common; that is, if x is in S_1 then x is not in S_2.

Solution. Denote the integers as $x_1, x_2, x_3, \ldots, x_{100}$, and consider the subsets $S_1 = \{x_1\}, S_2 = \{x_1, x_2\}, S_3 = \{x_1, x_2, x_3\}, \ldots, S_100 = \{x_1, x_2, \ldots, x_{100}\}$. A number is divisible by 100 if and only if it is equivalent to 0 modulo 100. We have two cases to consider:

Case 1: If one the sum of the integers in S_i is divisible by 100, then we can choose S_i as our desired subset.

Case 2: Otherwise, each of the 100 subsets is equivalent to one of $\{1, 2, 3, \ldots, 99\}$ modulo 100. If we take these 99 options to be our holes, and the 100 subsets to be our pigeons, then by the pigeonhole principle, we get that the sums of the integers in two of the sets are equivalent modulo 100. That is, for some indices i and j, where $i < j$, we have that

$$x_1 + x_2 + \cdots + x_i \equiv x_1 + \cdots + x_i + x_{i+1} \cdots + x_j \mod 100.$$

But this means that

$$(x_1 + \cdots + x_i + x_{i+1} \cdots + x_j) - (x_1 + x_2 + \cdots + x_i) \equiv 0 \mod 100$$

and hence that

$$x_i + x_{i+1} \cdots + x_j \equiv 0 \mod 100.$$

So, we can choose $\{x_i, x_{i+1}, \ldots, x_j\}$ to be our desired subset. ∎

5. Suppose that the numbers $0, 1, 2, \ldots, 9$ are randomly assigned to the vertices of a decagon.[2] Show that there are three consecutive vertices whose sum is at least 14.

Solution. Let x_0 be the number assigned to vertex 0, x_1 be the number assigned to vertex 1, x_2 be the number assigned to vertex 3, etc. Then, we need to prove that one of the sums $S_0 = x_0 + x_1 + x_2, S_1 = x_1 + x_2 + x_3, S_2 = x_2 + x_3 + x_4, \ldots, S_9 = x_9 + x_0 + x_1$ is at least 14. Notice that

$$S_0 + S_1 + \cdots + S_9 = 3(x_1 + x_2 + \cdots + x_9) = 3(0 + 1 + \cdots + 9) = 135$$

We can let our sums be our holes, and our total sum be our pigeons. So, we have 10 holes and 135 pigeons, and by the generalized pigeonhole principle, we must have at least one hole with $\lfloor \frac{135}{10} \rfloor = 14$ pigeons in it. So, one of our sums must be at least 14. ∎

.

6. Let S be a set of 3 distinct integers. Show that one can always choose two of them (say, a and b) such that $ab(a - b)(a + b)$ is divisible by 10.

Solution. A number is divisible by 10 if and only if it is divisible by 2 and 5. So, it suffices to show that we can pick a and b from S such that $a^3b - ab^3$ is divisible by 2 and 5.

Divisibility by 2: If any one of the factors ab, $a - b$, or $a + b$ is even, then $ab(a - b)(a + b)$ will be even and hence divisible by 2. If at least one of a or b is even, then ab will be even, and so the entire product will be even. If both a and b are odd, then $a - b$ will be even (so will $a + b$), and so again, the entire product will be even. So, regardless of the choice of a and b, it will always be the case that $ab(a - b)(a + b)$ is divisible by 2.

Divisibility by 5: We will consider 2 cases.

Case 1: Suppose that one of the three integers in S is divisible by 5. Choose this to be a, and pick either of the remaining integers to be b. Then, since a is divisible by 5, then the product $ab(a - b)(a + b)$ will also be divisible by 5, which is what we want.

Case 2: Otherwise, it must be the case that none of the three integers in S are equivalent to 0 modulo 5. So, we get that each must be equivalent to 1, 2, 3, or 4 modulo 5.

[2] A *decagon* is a polygon with 10 vertices.

Case 2a: If two of the three integers in S are equivalent modulo 5, then take these two integers to be a and b. Then, since $a \equiv b \mod 5$, we have that $5 \mid a - b$, and so the product $ab(a-b)(a+b)$ is divisible by 5.

Case 2b: Otherwise, all three integers in S bust have different values modulo 5. Consider the pairs $\{2, 3\}$ and $\{1, 4\}$. If we let these pairs be our holes and the three integers in S be our pigeons, then by the pigeonhole principle, we are either able to choose a and b such that $a \equiv 2 \mod 5$ and $b \equiv 3 \mod 5$ or $a \equiv 1 \mod 5$ and $b \equiv 4 \mod 5$. In both cases, we have that $a + b \equiv 0 \mod 5$, and hence that $a + b$ is divisible by 5. So, in both cases, the product $ab(a-b)(a+b)$ is divisible by 5. ■

7. (HARD)
   Show that any positive integer x containing N digits, none of which are 0, is either divisible by N or can be converted into an integer that is divisible by N by replacing some, but not all, of its digits with 0.

   Solution. Let Row 1 and Row 2 be two rows of N digits, where Row 1 consists of the digits of x, in order, and Row 2 consists of all 0's. Suppose we play a game where on each turn, we swap a nonzero digit of Row 1 with the corresponding 0-digit of Row 2. After i turns, exactly i digits of Row 1 will be 0. Let N_i denote the integer given by the digits of Row 2 after turn i. Furthermore, associate each turn with a state, where $state_i$ is the value of $N_i \mod N$ after turn i.

   Notice that we can take a maximum of N turns, since Row 2 has a total of N digits. Also, there are N possible states, since there are N possible values that an integer can have modulo N. We have two cases to consider:

   Case 1: If every turn results in a different state, then there must be some i such that $N_i \equiv 0 \mod N$. So, N_i is divisible by N, so N_i is the integer that we are looking for.

   Case 2: Otherwise, we have a total of N turns, and each turn takes on one of $N-1$ states. So, if we let the states be our holes and our turns be our pigeons, then by the pigeonhole principle, we must have turns i and j (without loss of generality, assume $i < j$) such that $state_i = state_j$. So, for these values of i and j, we have that $N_i \equiv N_j \mod N$. Now, define a third row, Row 3, that has the same function as Row 2, but only starts playing the game on turn $i+1$. That is, on turn $i+1$, Row 3 will have exactly one nonzero digit, and that digit corresponds to the digit that was swapped between Row 1 and Row 2 on turn $i+1$. Let N'_k denote the integer given by the digits of Row 3 after turn k. Since $N_i + N'_j = N_j$ and $N_i \equiv N_j \mod N$, we get that $N_i \equiv N_i + N'_j \mod N$ and hence that $N'_j \equiv 0 \mod N$. So, N'_j is divisible by N. Since N'_j has at least one nonzero digit, and all its nonzero digits are obtained by swapping a zero digit with a nonzero digit of N, then this is the integer we are looking for.

   ■