Grade 6 Math Circles

Nov 15/16/17, 2022

Geometric Constructions - Lesson

> "Geometry will draw the soul toward truth and create the spirit of philosophy."
>
> - Plato

> "There is geometry in the humming of the strings, there is music in the spacing of the spheres."
>
> - Pythagoras

Geometry in Ancient Greece

The influences of Ancient Greece are still being felt in our lives today. Democracy, deductive reasoning, and the Olympics are all elements of our lives that have their roots in Ancient Greece. Similarly, some parts of mathematics have roots in the work of Greek mathematicians such as Euclid, Archimedes, or Pythagoras. For example, Euclid's *Elements*, which establishes the fundamentals of geometry, is arguably one of the most influential (and beautiful) works of science in history.

What distinguishes Ancient Greek Mathematics is that it features very few numbers. Mathematicians at the time were more interested in shapes and diagrams, and interpreted numbers using lengths. Much of the math at the time featured **geometric constructions**, diagrams which were drawn using only a straightedge and a compass. Due to their prominence in Euclid's *Elements*, these also called **euclidean constructions** or **greek constructions**.

Rules of Geometric Constructions

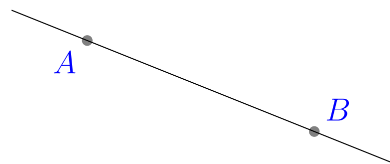
The rules of creating a geometric construction are simple: to draw a diagram, we can use a straightedge and a compass, but nothing else.

- Using a straightedge: Given any two points, we can draw a line passing through those two points. Note this **is not a ruler** since it has no measurements of length (but you can use a ruler as a straightedge, as long as you don't use the length measurements).
- Using a compass: Recall that a compass is a tool we use to draw circles. Given a point, and a length, we can draw a circle centered around that point with radius equal to that length.

While it may seem illogical to limit ourselves to only using these two tools (why not rulers? protractors? dividers?), there is a large range of geometric figures we can draw with these two tools.

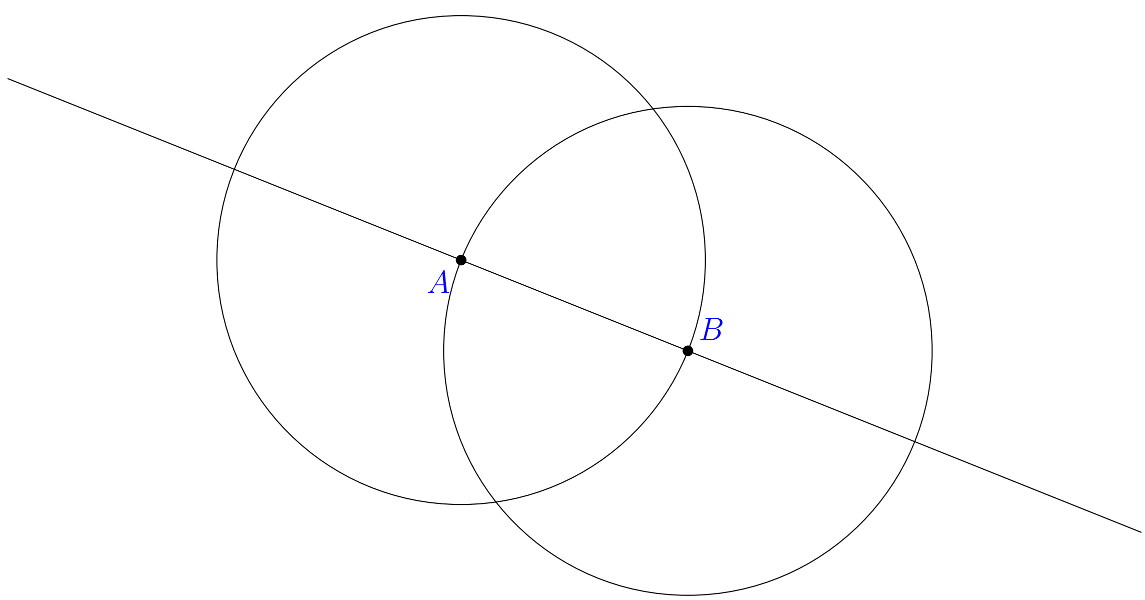
Introduction to Constructions

Let's first work through a simple example so you can get a sense of how constructions work. Let's first choose two points, call them A and B, and draw a line that passes through A and B:



Recall that a **line segment** is a part of a line which lies between two **endpoints**. For example, we can consider the line segment between A and B, which we write as $\overline{AB}$. When referring to the length of $\overline{AB}$ (the distance between A and B), we write AB (without the line on top).

Next draw two circles: one with center A and radius AB, and the other with center B and radius AB:



What happens when we draw a line through the two points where the circle intersect?

The **perpendicular bisector** of a line segment $\overline{AB}$ is the line which bisects $\overline{AB}$ at a right angle. That is, it intersects $\overline{AB}$ at its midpoint, and is perpendicular to $\overline{AB}$.

In order to make sure we can replicate our construction, we can summarize our steps.

Example 1

Given a line segment $\overline{AB}$, construct the perpendicular bisector of $\overline{AB}$.

Example 1 Solution

1. Construct the circle with center A and radius AB.
2. Construct the circle with center B and radius BA.
3. Draw the line which passes through the intersection points of the two circles. This line is the perpendicular bisector of $\overline{AB}$.

Note: we can keep track of how many steps our construction takes. In some cases (not in this lesson), we may want our construction to have as few steps as possible.

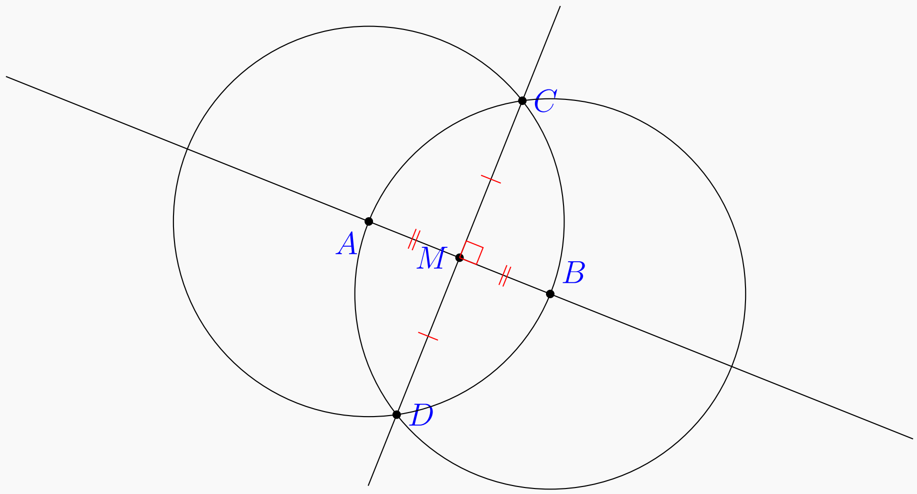
Exercise 1

Using the construction for a perpendicular bisector (or otherwise), construct:

- A right triangle.
- An equilateral triangle.
- A parallelogram.
- A rectangle.

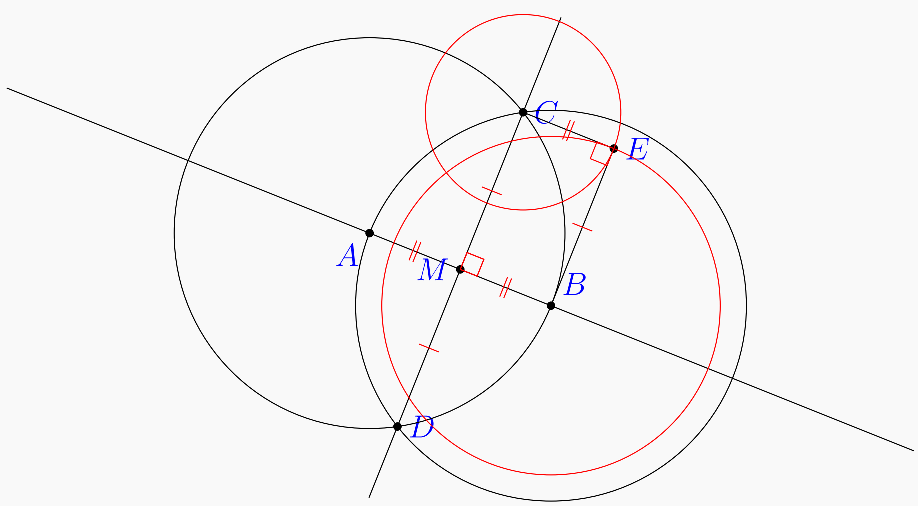
Exercise 1 Solution

Let's start with our construction for a perpendicular bisector. Let M be the midpoint of $\overline{AB}$ (the point on $\overline{AB}$ such that $AM = MB$) and let C and D be the two points where the circles intersect.



- $\triangle BMC$ is a right triangle since BM is perpendicular to CM.
- $\triangle ABC$ is equilateral since $AC = AB$ and $BC = AB$ (by construction of C).
- Quadrilateral $ADBC$ is a parallelogram (by symmetry).
- We will construct a rectangle using vertices B, C and M. Let E be the fourth vertex. Then E will satisfy $CE = BM$ and $BE = CM$. Thus,
  1. Construct a circle with center C and radius BM.
  2. Construct a second circle with center B and radius CM.
  3. Let E be the intersection of the intersection of these two circles (such that CE and BM don't intersect).

  Then $CE = BM$ and $BE = CM$. Therefore, $BMCE$ will be a rectangle:

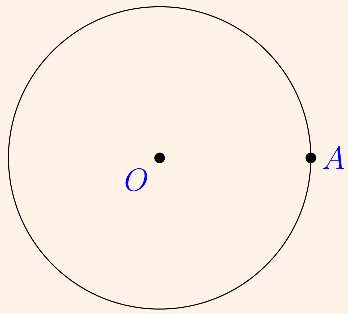


Example 2

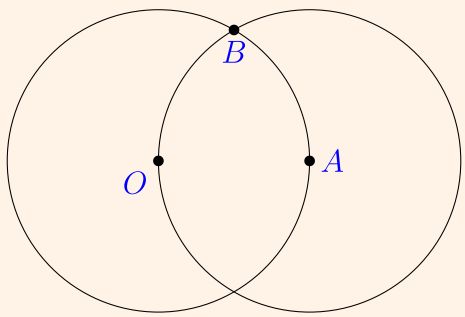
Construct a regular hexagon.

Example 2 Solution

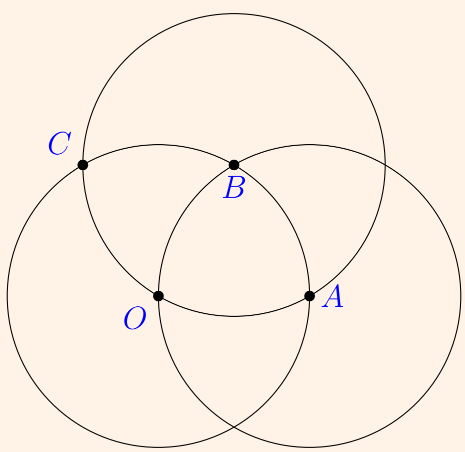
Note we can do this by repeatedly creating equilateral triangles, since a regular hexagon consists of 6 equilateral triangles. However, there is a much nicer way of construction a regular hexagon. First, draw with a circle with center O and let A be a point on the circle.



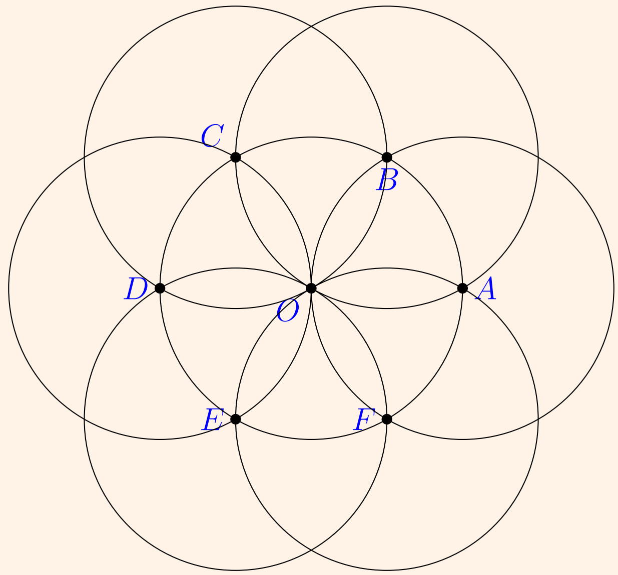
Next, construct a circle with center A and radius AO:



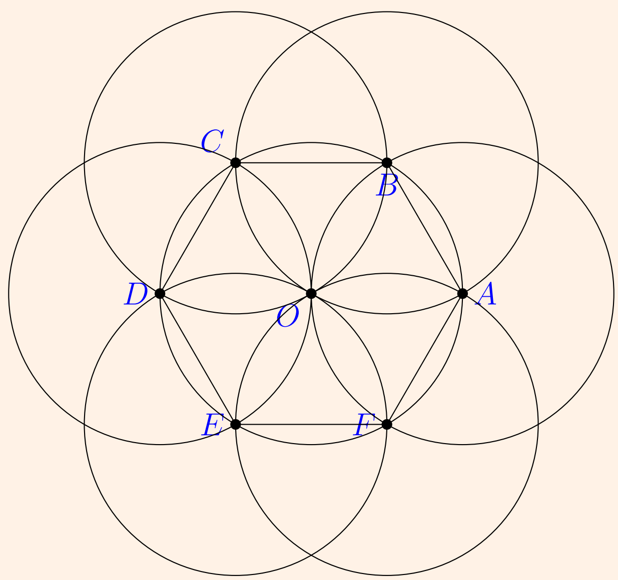
This new circle intersects the original one at point B. We can once again draw a circle with center B and radius BO:



Let the intersection of this new circle and the original one be C. We can continue in this fashion: draw a circle with center C and radius CO, which intersects the first circle at D; etc. until we get back to A:



And we can now finish our construction because $ABCDEF$ is a regular hexagon:

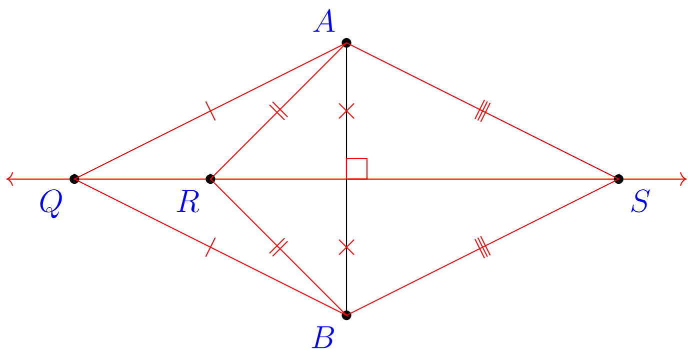


Stop and Think

Why is $ABCDEF$ a regular hexagon?

Perpendicular Bisectors in Triangles

A perpendicular bisector of a line segment $\overline{AB}$ can also be defined as the set of points which are of equal distance from A and B:



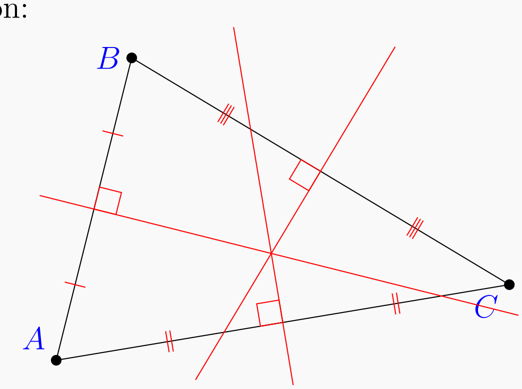
For example, $QA = QB$, $RA = RB$, $SA = SB$, and for any other point P on the perpendicular bisector of $\overline{AB}$, $PA = PB$ (given any point P, you can show $PA = PB$ using your compass).

Exercise 2 - Part 1

Construct a triangle (pick any three points, the triangle does not have to be equilateral). Construct the perpendicular bisectors of all three sides. What do you notice?

Exercise 2 - Part 1 Solution

Since we've already worked through the construction of a perpendicular bisector, we will just provide the final construction:



The three perpendicular bisectors intersect! When three lines intersect, we say that the three lines are **concurrent**.

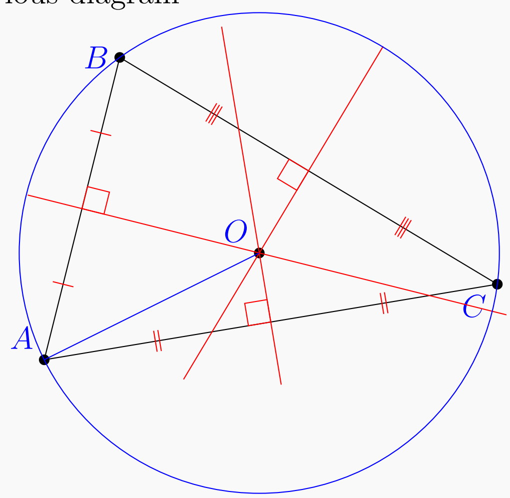
In any triangle, the perpendicular bisectors of the three sides will intersect at one point. We call this point the **circumcenter**.

Exercise 2 Part 2

In the triangle you constructed in part 1, let O be the circumcenter and A be one of the vertices. Construct a circle with center O and radius OA. What do you notice?

Exercise 2 - Part 2 Solution

Adding the circle to our previous diagram



By definition, O is on the perpendicular bisector of $\overline{AB}$ therefore $OA = OB$. Similarly, O is on the perpendicular bisector of $\overline{AC}$ therefore $OA = OC$. Thus, A, B, and C are all on the circle with center O and radius OA.

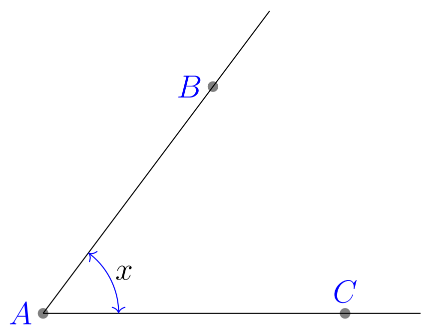
For every triangle, there is a unique circle which passes through each vertex. We call this circle the **circumcircle**. Its center is the **circumcenter**, the intersection of the three perpendicular bisectors.

Stop and Think

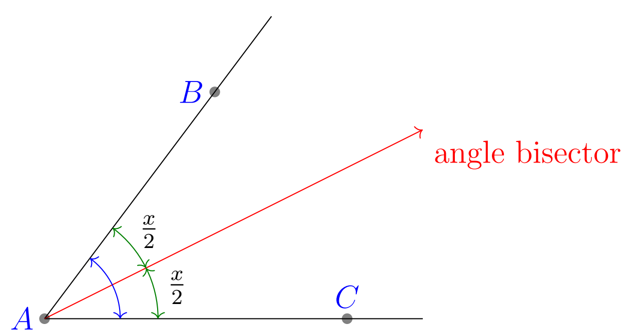
Does every quadrilateral have a circle which passes through each vertex?

Angle Bisector

The **angle bisector** of an angle is the line which divides the angle into two angles of equal measure (it divides the angle in half). For example, given an angle of measure x:



The angle bisector would be

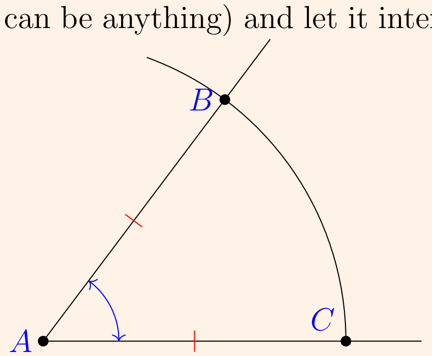


Example 3

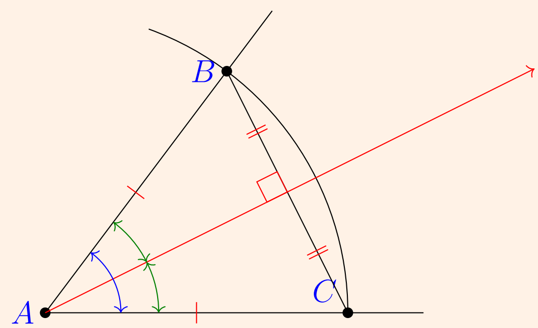
Given an angle, construct its angle bisector.

Example 3 Solution

Suppose we have an an angle formed by two lines which intersect at a point A. First, draw a circle with center A (the radius can be anything) and let it intersect the two lines at B and C.



Next, construct the perpendicular bisector of BC. We can do so by constructing the two circles with centers B and C, and radius BC:



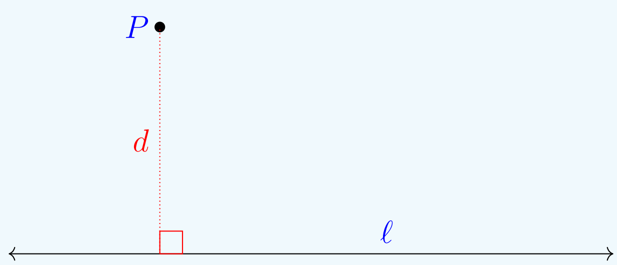
This line will be the angle bisector of $\angle BAC$.
Why? The triangle ABC is isosceles since $AB = AC$. By symmetry, $\angle ABC = \angle ACB$ therefore we can argue that the angle bisector of $\angle A$ must be the same as the perpendicular bisector of BC.

Fun Fact

It is not possible to construct two lines which trisect an angle (divide it into 3 equal angles).

We the use the word **equidistant** to mean 'equally distant'. For example, if $\overline{AB}$ is a line segment, then all points on the perpendicular bisector of $\overline{AB}$ are equidistant from A and B.
We can provide a similar observation for angle bisectors:

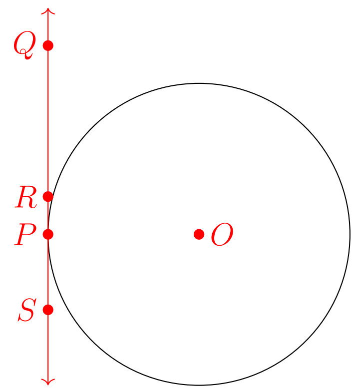
The distance from a point P to a line ℓ is defined as the shortest distance from P to any point on ℓ. When P is not on ℓ, we can illustrate the distance d, from P to ℓ, as:



All points on an angle bisector are **equidistant** from the two lines which form the angle.

Tangents to a Circle

A line that is **tangent** to a circle is a line which intersects the circle at exactly one point. For example, the red line in the figure below is tangent at P to the circle with center O and radius OP.

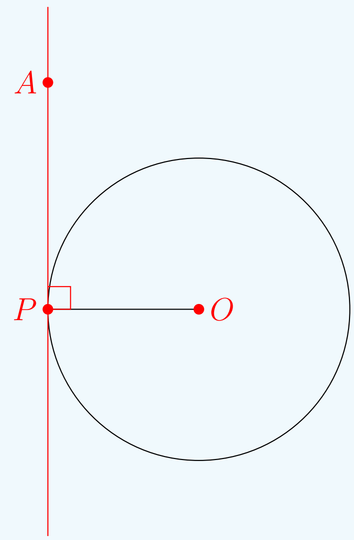


Stop and Think

In the above diagram, draw the line segments OP, OQ, OR, and OS. Estimate the angles which $\overline{OP}$, $\overline{OQ}$, $\overline{OR}$, and $\overline{OS}$ form with the red line.

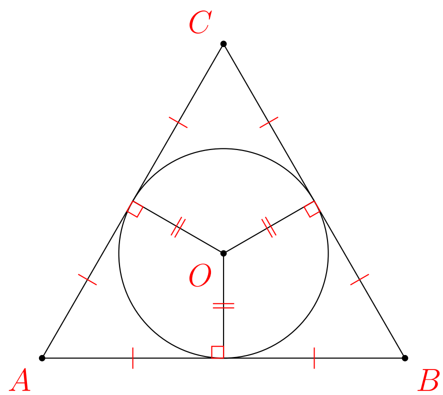
Important Fact about Tangents

If $\mathcal{C}$ is a circle with center O and ℓ is a line that is tangent to $\mathcal{C}$ at a point P. Then, if A is a point on ℓ, $\angle APO = 90°$:



Inscribed Circles

A circle is **inscribed** in a polygon (triangle, quadrilateral, etc.) if each side of the polygon is tangent to the circle. For example, here's a circle inscribed in an equilateral triangle ABC:



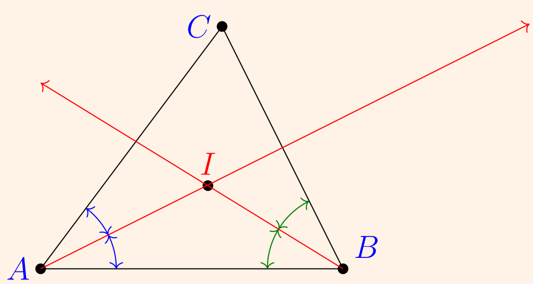
Visually, an inscribed circle is inscribed in a polygon if it is exactly contained by that polygon. If a circle is inscribed in a polygon, then its center is **equidistant** from each side. That is, the distance from the center to each side is the same (equal to the radius of the circle).
Therefore, the center must lie on the angle bisector of each interior angle!

Example 4

Given a triangle, construct a circle which is inscribed in the triangle.

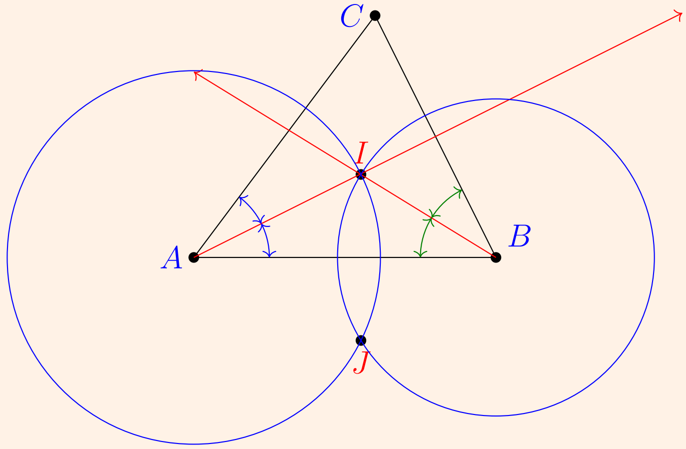
Example 4 Solution

As mentioned above, we can construct this circle by finding the intersection of the angle bisectors. Given a triangle ABC, first construct the angle bisectors of $\angle ABC$ and $\angle BAC$. Let them intersect at I.

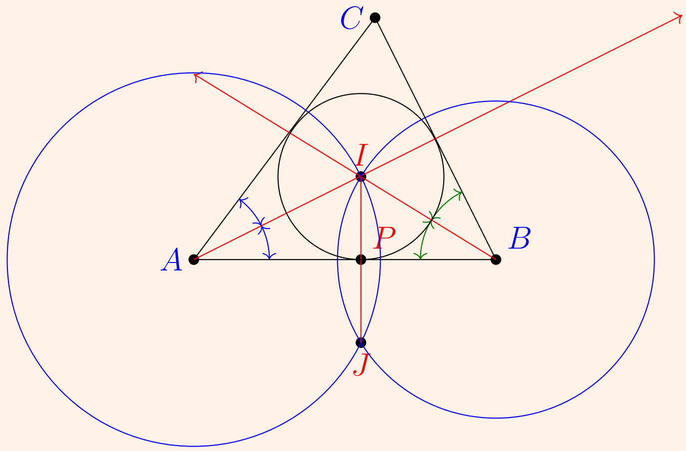


We know that I will be the center of the circle we wish to construct, it now remains to find the radius. To do so, we need to find the distance from I to line AB. That is, we need to construct a line passing through I which is perpendicular to AB. We can do this in the following way:

1. Construct the circle with center A and radius AI.
2. Construct the circle with center B and radius BI.
3. Let the second intersection of these two circles be J (the first one is I).



4. Let $\overline{JI}$ intersect $\overline{AB}$ at P. Then the circle with center I and radius IP will be inscribed in triangle ABC.



Given a triangle ABC, there is a unique circle which is inscribed in ABC. We call this circle the **incircle**. Its center is called the **incenter** and is the intersection of the three angle bisectors (of the interior angles) of the triangle.