Grade 7/8 Math Circles

November 14/15/16/17, 2022

Modular Arithmetic Problem Set

For an extra challenge try to complete all the problems without using a calculator.

1. Fill in the blank with either $\equiv$ or $\not\equiv$.

   a) -3 ___ $7 \pmod 5$
   b) -10 ___ $12 \pmod{11}$
   c) 3 ___ $10 \pmod 4$
   d) 7 ___ $-43 \pmod 2$
   e) 8 ___ $104 \pmod 8$
   f) 89 ___ $1 \pmod 9$

2. Fill in the blank with either $\equiv$ or $\not\equiv$.

   a) 2 ___ $47 \pmod 1$
   b) -2448 ___ $39202 \pmod 1$
   c) a ___ $b \pmod 1$ for any integers a and b

3. List three integers which are congruent to k modulo m given the following values for k and m.

   a) $k = 9$, $m = 21$
   b) $k = -19$, $m = 3$
   c) $k = 87$, $m = 7$
   d) $k = 0$, $m = 15$
   e) $k = -2$, $m = 2$
   f) $k = -11$, $m = 8$

4. What is the remainder when...

   a) 320×84^7 is divided by 3?
   b) 22928^3 is divided by 5?
   c) 17^{404} is divided by 15?
   d) $97 - 106^2 + 100^{429}$ is divided by 9?
   e) $(19 \times 239) + 282^6$ is divided by 20?
   f) $1430 \times (11 + 153^{200064})$ is divided by 14?

5. What is the last digit of $(391^{283} + 28917 - 283^4) \times 85$?

6. Gus baked 10 trays of muffins to share with his class. Each tray contained 12 muffins. Gus split the muffins evenly between his 22 classmates and his teacher and then he ate the leftovers. How many muffins did Gus eat?

7. A card game involves removing one random card from a regular deck of 52 cards by each player Then, the remaining cards are dealt between all the players. For each of the following number of players, does each player have the same amount of cards?

   a) 2 players   b) 3 players   c) 5 players   d) 13 players

8. Morgan, Milly, and Baloo are playing a game and need to decide who will go first. They decide that the decision should be random, so the numbers 0, 1, and 2 are all uniquely assigned to the three of them and each of them will hold up a random number of fingers (from 0 to 10) at the same time. Whoever's assigned number is congruent modulo 3 to the total number of fingers held up gets to go first.

   Who will go first if Morgan is assigned 0 and holds up 6 fingers, Milly is assigned 1 and holds up 1 finger, and Baloo is assigned 2 and holds up 9 fingers?

9. What day of the week were you born on? Work backwards from your last or next birthday.

   Hint: The years 2020, 2016, 2012, 2008, 2004, and 2000 were all leap years, that is, these years contained 366 days instead of 365.

   Once you've found the day of the week, check your answer by looking at a calendar from your birth year.

Congruence Classes

Two integers are congruent modulo m if they have the same remainder when divided by a positive integer m. Is there a way to define all integers that are congruent modulo m?

Congruence classes are sets of integers which have the same remainder when divided by a positive integer m. That is, all integers in a congruence class are congruent modulo m. We denote a congruence class as $[a]$ where a is the remainder.

Recall that the remainder of a division, using divisor n, is an integer between 0 and $n-1$. So, congruence classes for modulus m only exist from 0 to $m-1$.

Example

We can organize integers into congruence classes with modulus m by finding which integer from 0 to $m-1$ each integer is congruent with. For example, with modulus 5, we can write the congruences for the integers 1 to 14 as seen below:

$0 \equiv 0 \pmod 5$	$5 \equiv 0 \pmod 5$	$10 \equiv 0 \pmod 5$
$1 \equiv 1 \pmod 5$	$6 \equiv 1 \pmod 5$	$11 \equiv 1 \pmod 5$
$2 \equiv 2 \pmod 5$	$7 \equiv 2 \pmod 5$	$12 \equiv 2 \pmod 5$
$3 \equiv 3 \pmod 5$	$8 \equiv 3 \pmod 5$	$13 \equiv 3 \pmod 5$
$4 \equiv 4 \pmod 5$	$9 \equiv 4 \pmod 5$	$14 \equiv 4 \pmod 5$

Using these congruences, we can organize the integers 1-14 into their respective congruence classes.

- 0, 5, and 10 belong to the same congruence class, $[0] = \{..., -15, -10, -5, 0, 5, 10, 15, ...\}$
- 1, 6, and 11 belong to the same congruence class, $[1] = \{..., -14, -9, -4, 1, 6, 11, 16, ...\}$
- 2, 7, and 12 belong to the same congruence class, $[2] = \{..., -13, -8, -3, 2, 7, 12, 17, ...\}$
- 3, 8, and 13 belong to the same congruence class, $[3] = \{..., -12, -7, -2, 3, 8, 13, 18, ...\}$
- 4, 9, and 14 belong to the same congruence class, $[4] = \{..., -11, -6, -1, 4, 9, 14, 19, ...\}$

10. How many congruence classes are there for each modulus?

    a) 1 b) 2 c) 3 d) n where n is any positive integer

11. What congruence classes exist for the following moduli (plural of modulus)? List 3 numbers that belong to each congruence class.

    a) 3 b) 6 c) 1 d) 2

Modular Addition with Congruence Classes

The basic idea of modular addition is that we are just adding the remainders while multiples of the modulus are ignored. And since congruence classes are defined by their remainders, we can see that if we have two congruence classes, then the sum of two integers where one is from each of the two congruence classes will always be the same no matter which integers were picked from the congruence classes.

Modular Addition with Congruence Classes

Suppose that a and b are integers and that $[a]$ and $[b]$ are two congruence classes with modulus m. We can define modular addition as $[a] + [b] = [s]$ where $a + b \equiv s \pmod{m}$ and $0 \leq s < m$.

We can now build addition tables that will provide us with all the addition results between any congruence classes with a specific modulus. For example, to the right is an addition table for all congruence classes with modulus 7.

+	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[0]
[2]	[2]	[3]	[4]	[5]	[6]	[0]	[1]
[3]	[3]	[4]	[5]	[6]	[0]	[1]	[2]
[4]	[4]	[5]	[6]	[0]	[1]	[2]	[3]
[5]	[5]	[6]	[0]	[1]	[2]	[3]	[4]
[6]	[6]	[0]	[1]	[2]	[3]	[4]	[5]

Example

Since $11 \equiv 4 \pmod{7}$ and $8 \equiv 1 \pmod{7}$, $11 + 8 \equiv 4 + 1 \equiv 5 \pmod{7}$.
Similarly, since $18 \equiv 4 \pmod{7}$ and $78 \equiv 1 \pmod{7}$, $18 + 78 \equiv 4 + 1 \equiv 5 \pmod{7}$.

Both 11 and 18 belong to the congruence class [4] and both 8 and 78 belong to the congruence class [1] with modulus 7.

Notice that $11 + 8$ and $18 + 78$ are both congruent to 5 modulo 7. Looking at our above addition table for modulus 7, we can see that $[4] + [1] = [5]$ by finding the row and column intersections as highlighted in the table. This means that the sum of any integer which is congruent to 4 modulo 7 and any integer which is congruent to 1 modulo 7 will be congruent to 5 modulo 7.

12. Perform the following additions on the congruence classes with modulus 7 using the above addition table.

a) $[3] + [5]$ b) $[5] + [3]$ c) $[2] + [2]$ d) $[6] + [1]$ e) $[0] + [4]$

Modular Multiplication with Congruence Classes

The basic idea of modular multiplication is that we are just multiplying the remainders and multiples of the modulus are ignored. And since congruence classes are defined by their remainders, we can see that if we have two congruence classes, then the product of two integers where one is from each of the two congruence classes will always be the same no matter which integers were picked from the congruence classes.

Modular Multiplication with Congruence Classes

Suppose that a and b are integers and that $[a]$ and $[b]$ are two congruence classes with modulus m. We can define modular multiplication as $[a] \times [b] = [s]$ where $a \times b \equiv s \pmod{m}$ and $0 \leq s < m$.

We can now build multiplication tables that will provide us with all the multiplication results between any congruence classes with a specific modulus. For example, to the right is a multiplication table for all congruence classes with modulus 7.

×	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[0]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[0]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[0]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[0]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[0]	[6]	[5]	[4]	[3]	[2]	[1]

Example

Since $11 \equiv 4 \pmod 7$ and $8 \equiv 1 \pmod 7$, $11 \times 8 \equiv 4 \times 1 \equiv 4 \pmod 7$.
Similarly, since $18 \equiv 4 \pmod 7$ and $78 \equiv 1 \pmod 7$, $18 \times 78 \equiv 4 \times 1 \equiv 4 \pmod 7$.

Both 11 and 18 belong to the congruence class [4] and both 8 and 78 belong to the congruence class [1] with modulus 7.

Notice that 11×8 and 18×78 are both congruent to 4 modulo 7. Looking at our above multiplication table for modulus 7, we can see that $[4] \times [1] = [4]$ by finding the row and column intersections as highlighted in the table. This means that the product of any integer which is congruent to 4 modulo 7 and any integer which is congruent to 1 modulo 7 will be congruent to 4 modulo 7.

13. Perform the following multiplications on the congruence classes with modulus 7 using the above multiplication table.

a) $[2] \times [5]$ b) $[5] \times [2]$ c) $[3] \times [3]$ d) $[6] \times [1]$ e) $[0] \times [4]$