Math Circles - Pigeonhole Principle - Fall 2022

Exercises

Graph Theory

1. Show that in a group of 100 people, at least two people must know the same number of people (assuming that knowing each other is mutual, i.e., if I know you then you also know me).

   Solution. We can represent this situation as a graph G, where the people are the vertices, and an edge between vertex A and vertex B means that Person A and Person B know each other. The number of people that Person A knows is the number of vertices vertex A is connected to via an edge, which is the degree of vertex A. So, this problem is an instance of the first exercise from class. For completeness, I will rewrite the proof below for the case $n = 100$.

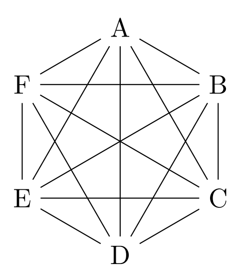
   Since G has 100 vertices, any given vertex can be connected to anywhere from 0 to 99 other vertices via an edge. So, the degree of each vertex of G could be any integer from 0 to 99. So, there are 100 possibilities for the degree of a vertex in G.

   If each of the 100 vertices had a different degree, then we would have exactly one vertex with each possible degree. In particular, one vertex would have degree 99, and therefore would be connected to every other vertex. This includes the vertex with degree 0. But this is impossible, because a vertex with degree 0 cannot be connected to any other vertex.

   So, we cannot have both a vertex of degree 0, and a vertex of degree $n-1$. In either case, this leaves 99 options for the degree of each vertex. If we let these 99 options be our holes, and the 100 vertices be our pigeons, then by the pigeonhole principle, two vertices must have the same degree. ■

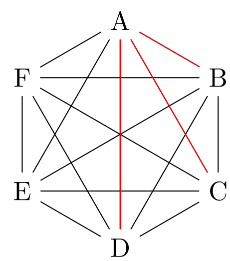
2. Prove that, in a group of 6 children (all of whom know each other), there will always either be a group of 3 children who are all friends, or a group of 3 children who do not get along.

   Solution. We can represent this scenario as a graph G with 6 vertices, where each vertex represents a child. If Child A and Child B are friends, then we put a red edge between vertex A and vertex B; otherwise, we put a blue edge between them. Notice that every vertex is connected to every other vertex by an edge of some colour, so (ignoring the colours of the edges), G looks like this:

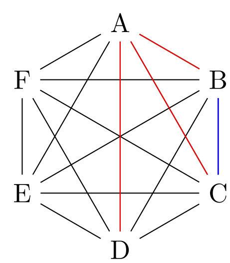


If a group of three children are all friends, then G will have a red triangle. If a group of three children do not get along with each other, then G will have a blue triangle. So, we need to show that no matter how we colour the edges of G, it will either have a red triangle or a blue triangle.

Without loss of generality, consider vertex A. Vertex A has 5 and each of these edges is either red or blue. If we let the edges be our pigeons and the colours red and blue be our holes, then by the generalized pigeonhole principle, at least 3 of these edges must be the same colour. So, without loss of generality, we can suppose that the edges AB, AC, and AD are all red.



If either BC or CD are red, then we have a red triangle, and we are done. So, assume that they are blue.



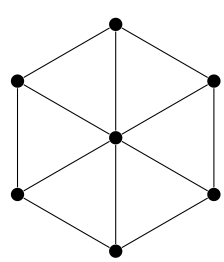
Now, consider the edge BD. If BD is red, then we have a red triangle ABD. If BD is blue, then we have a blue triangle BCD. So, we are guaranteed a monochromatic triangle.

■

Colouring the Plane

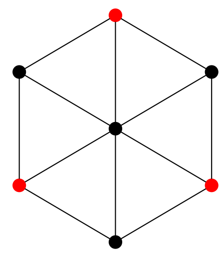
1. Colour the plane with two colours. Prove that there must be an equilateral triangle whose vertices are all the same colour.

   Solution. Suppose the two colours we use are red and blue. Consider 6 points in the plane as follows, where each line has equal length (i.e., each of the smaller triangles is equilateral):

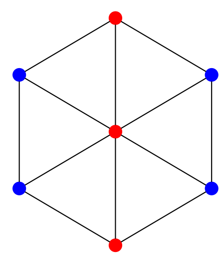
   

   Without loss of generality, suppose that the middle vertex is red. If we let the outer vertices be our pigeons and our pigeons and the colours red and blue be our holes, then by the pigeonhole principle, we get that at least 3 of the outer vertices are the same colour.

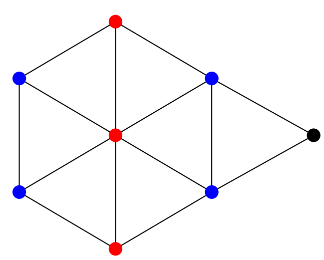
If these three vertices are all non-adjacent (like this),



then those three form an equilateral triangle, regardless of whether the are red or blue. If that is not the case, then we have two adjacent outer vertices of the same colour. If these vertices are red, then together with the middle vertex, they form an equilateral triangle. Otherwise, we must have the following situation:



Add another node, as follows:



If the new node is blue, then we have a blue triangle. If it is red, we have a red triangle. So, we are done.

So, in all cases, we end up with a monochromatic equilateral triangle. ■

2. Colour the plane with two colours. Prove that an any line segment, there must be three points of the same colour such that one point is exactly halfway in between the other two points.

   Solution. Consider the points on the x-axis; that is, points of the form $(x, 0)$. Let $(a, 0)$ and $(b, 0)$ be distinct points of the same colour (without loss of generality, suppose they are red). If either of the points $(2a - b, 0)$ and $(2b - a, 0)$ are red, then we are done. If not, then it must be the case that both $(2a - b, 0)$ and $(2b - a, 0)$ are blue. Now, consider the point $(\frac{a+b}{2}, 0)$. This is the midpoint of the line segment with endpoints $(a, 0)$ and $(b, 0)$, so if this point is red, then we are done. But it is also the midpoint of the line segment with endpoints $(2a - b, 0)$ and $(2b - a, 0)$, so if it is blue, then we are also done.

So, in all cases, we can find three points of the same colour such that one point is the midpoint of the line segment between the other two points.

**Note that we didn't explicitly use the pigeonhole principle here, although it still had somewhat of a pigeonhole-y flavour. ■

3. Colour the plane with n colours. Prove that one can find a rectangle whose four corners are all the same colour.

   Solution. Consider the positive lattice points in the plane; that is, consider the set of all points whose x- and y-coordinates are both positive integers. Notice that the points (a, b), $(a + c, b)$, $(a, b + d)$, $(a + c, b + d)$ form a rectangle. If we can show that there must be 4 positive lattice points of this form which are all the same colour, then we win.

   First, consider how long our "rows" need to be. In order to form a rectangle, we will need each row to have at least two points of the same colour in it. Since there are n different colours, then if we let the colours be our holes and each lattice point on a given row to be our pigeons, then by the pigeonhole principle, we get that we need our rows to each contain at least $n+1$ lattice points in order to guarantee that each row has two of the same colour in it.

   Now, consider how long our "columns" need to be. In order to form a rectangle, we need to guarantee that we will have two rows whose same-colour points are the same colour in both rows, and that those two points are in the same position in both rows. Since our rows each contain $n + 1$ lattice points, and there are n possible colours, there are n^{n+1} different possible ways to colour a row, since there are n options for each colour in the row. So, if we let the different colour arrangements be our holes, and the number of rows be our pigeons, then by the pigeonhole principle, we need $n^{n+1} + 1$ rows in order to guarantee that two rows are coloured in the exact same way. But if two rows are coloured in the exact same way, then in particular, the two same-colour points in both rows must be the same colour in both rows, and in the same position.

   So, if we look at the positive lattice points whose x-coordinates range from 0 to $n - 1$ and whose y-coordinates range from 0 to $n^{n+1} + 1$, then somewhere inside there, we will have 4 of them which are all the same colour and form a rectangle. ■