Grade 7/8 Math Circles
November 21/22/23/24, 2022
Complex Numbers - Solution

1. Find an equivalent expression to the following using i:

   (a) $\sqrt{-81}$

   (b) $-\sqrt{-1}$

   (c) $-\sqrt{-12}$

   (d) $\sqrt{-\frac{10}{100}}$

   Solution:

   (a) $\sqrt{-81} = \sqrt{81}i = 9i$

   (b) $-\sqrt{-1} = -i$

   (c) $-\sqrt{-12} = -\sqrt{12}i = -\sqrt{2^2 \times 3}i = -\sqrt{2^2} \times \sqrt{3}i = -2\sqrt{3}i$

   (d) $\sqrt{-\frac{10}{100}} = \sqrt{\frac{10}{100}}i = \sqrt{\frac{10}{10^2}}i = \frac{\sqrt{10}}{\sqrt{10^2}}i = \frac{\sqrt{10}}{10}i$

2. Using the FOIL method, expand the following expressions:

   (a) $(a+2)(a+3)$

   (b) $(b+1)(c-1)$

   (c) $(2+3x)(1-2x)$

   (d) $(1+4i)(3+2i)$ (Simplify the expression using $i^2 = -1$.)

   Solution:

   (a) $(a+2)(a+3) = a^2 + 3a + 2a + 6 = a^2 + 5a + 6$

   (b) $(b+1)(c-1) = bc - b + c - 1$

   (c) $(2+3x)(1-2x) = 2 - 4x + 3x - 6x^2 = 2 - x - 6x^2$

   (d) $(1+4i)(3+2i) = 3 + 2i + 12i + 8i^2 = 3 + 14i - 8 = -5 + 14i$

3. Find the real part and the imaginary part of the following complex numbers:

   (a) $7 + 7i$

   (b) 900

   (c) $-100i + 12$

   (d) i

   (e) 0

Solution:

(a) The real part is 7 and the imaginary part is also 7.

(b) The real part is 900 and the imaginary part is 0.

(c) The real part is 12 and the imaginary part is -100.

(d) The real part is 0 and the imaginary part is 1.

(e) The real part is 0 and the imaginary part is 0.

4. Find the following complex numbers:

   (a) $(11 - 23i) + (7i - 17)$

   (b) $(8 - 11i) + 100i$

   (c) $99 - (3 + 77i)$

   (d) $(9i + 11) - (-5 + 7i)$

   (e) $(-3 + 2i)(1 - 5i)$

   (f) $(7i + 2)(i - 1)$

   (g) $\dfrac{5 + i}{1 - i}$

   (h) $\dfrac{4 - 5i}{4 + 5i}$

Solution:

(a) $(11 - 23i) + (7i - 17) = (11 + (-17)) + ((-23) + 7)i = -6 - 16i$

(b) $(8 - 11i) + 100i = 8 + ((-11) + 100)i = 8 + 89i$

(c) $99 - (3 + 77i) = (99 - 3) - 77i = 96 - 77i$

(d) $(9i + 11) - (-5 + 7i) = (11 - (-5)) + (9 - 7)i = 16 + 2i$

(e) $(-3+2i)(1-5i) = -3+15i+2i-10i^2 = -3+17i+10 = 7+17i$

(f) $(7i+2)(i-1) = 7i^2-7i+2i-2 = -7-5i-2 = -9-5i$

(g) $\dfrac{5+i}{1-i} = \dfrac{(5+i)(1+i)}{(1-i)(1+i)} = \dfrac{5+5i+i+i^2}{1+i-i-i^2} = \dfrac{5+6i-1}{1+1} = \dfrac{4+6i}{2} = 2+3i$

(h) $\dfrac{4-5i}{4+5i} = \dfrac{(4-5i)(4-5i)}{(4+5i)(4-5i)} = \dfrac{16-20i-20i+25i^2}{16-20i+20i-25i^2} = \dfrac{16-40i-25}{16+25} = \dfrac{-9-40i}{41}$

5. Find the complex conjugates of following complex numbers:

(a) 500

(b) $-777i$

(c) $1+7i$

(d) i

(e) 0

Solution:

(a) 500

(b) $777i$

(c) $1-7i$

(d) $-i$

(e) 0

6. Calculate the following powers of i:

   (a) i^{501}

   (b) $i^{44444444}$

   (c) i^{6}

   (d) $i^{10000003}$

Solution:

(a) $i^{501} = i^{4\times125+1} = i$

(b) $i^{44444444} = i^{4\times11111111} = 1$

(c) $i^{6} = i^{4+2} = i^{2} = -1$

(d) $i^{10000003} = i^{4\times2500000+3} = i^{3} = -i$

7. Using the FOIL method, calculate the following:

   (a) $(2-2i)^4$

   (b) $(i+1)^{10}$

   (c) $(3+3i)^4$

Solution:

(a) Using the FOIL method, we have $(2-2i)^2 = 4 - 8i + 4i^2 = -8i$, then

$$(2-2i)^4 = (2-2i)^2 \times (2-2i)^2 = (-8i) \times (-8i) = 64i^2 = -64$$

(b) Using the FOIL method, we have $(i+1)^2 = i^2 + 2i + 1 = 2i$, then

$$(i+1)^10 = [(i+10)^2]^5 = [2i]^5 = 32i^5 = 32i$$

(c) Using the FOIL method, we have $(3+3i)^2 = 9 + 18i + 9i^2 = 18i$, then

$$(3+3i)^4 = (3+3i)^2 \times (3+3i)^2 = 18i \times 18i = 324i^2 = -324$$

8. Find the solutions of the following quadratic equations:

   (a) $x^2 - 1 = 0$

   (b) $x^2 - 10x + 20 = -5$

   (c) $2x^2 - 5x + 2 = 0$

   (d) $x^2 + 7x + 1 = 0$

   (e) $x^2 + 4 = 0$

Solution:

(a) Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-0 \pm \sqrt{0^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{\pm\sqrt{4}}{2} \\ &= \pm 1 \\ \therefore x &= -1, 1 \end{aligned}$$

(b) By rearranging the equation to $ax^2 + bx + c = 0$ form, we get $x^2 - 10x + 25 = 0$. Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(25)}}{2(1)} \\ &= \frac{10 \pm \sqrt{100 - 100}}{2} \\ \therefore x &= 5 \end{aligned}$$

(c) Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(2)}}{2(2)} \\ &= \frac{5 \pm \sqrt{25 - 16}}{4} \\ &= \frac{5 \pm \sqrt{9}}{4} \\ &= \frac{5+3}{4}, \frac{5-3}{4} \\ \therefore x &= 2, \frac{1}{2} \end{aligned}$$

(d) Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4(1)(1)}}{2(1)} \\ &= \frac{-7 \pm \sqrt{49 - 4}}{2} \\ &= \frac{-7 \pm \sqrt{45}}{2} \\ &= \frac{-7 \pm 3\sqrt{5}}{2} \\ \therefore x &= \frac{-7+3\sqrt{5}}{2}, \frac{-7-3\sqrt{5}}{2} \end{aligned}$$

(e) Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-0 \pm \sqrt{0^2 - 4(1)(4)}}{2(1)} \\ &= \frac{\pm\sqrt{-16}}{2} \\ &= \frac{\pm 4i}{2} \\ &= \pm 2i \\ \therefore x &= -2i, 2i \end{aligned}$$

9. Verify that $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ is a 3rd root of one.

Solution: We will show that $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3=1$.

Using the FOIL method, we have

$$\begin{aligned}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2 &= \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\times\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\ &= \frac{1}{4}-\frac{\sqrt{3}}{4}i-\frac{\sqrt{3}}{4}i+\frac{3}{4}i^2\\ &= \frac{1}{4}-\frac{2\sqrt{3}}{4}i-\frac{3}{4}\\ &= -\frac{1}{2}-\frac{\sqrt{3}}{2}i\end{aligned}$$

Using the above information, we get

$$\begin{aligned}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 &= \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2\times\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\ &= \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\times\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\\ &= \frac{1}{4}-\frac{\sqrt{3}}{4}i+\frac{\sqrt{3}}{4}i-\frac{3}{4}i^2\\ &= \frac{1}{4}+\frac{3}{4}\\ &= 1.\end{aligned}$$

Therefore, $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ is a 3rd root of one.