Grade 7/8 Math Circles
October 2/3/4/5
Constructable Numbers

Math Like the Ancient Greeks

Over 2000 years ago, math all over the world looked very different than it does today. Familiar symbols like '+' and '=' did not exist, and everything was written using just words.

Example 1

To write out the equation $2 + 2 = 4$, the Ancient Greeks would write

Two added to two is equal to four

To write out the equation $3 \times 2 = 6$, the Ancient Greeks would write

Three multiplied by two is equal to six

Without the use of symbols, the Ancient Greeks performed almost all of their math using just geometry; they would draw out their equations, using shapes, and then reason what would be true.

Constructable Numbers

It's important to note that, although the Greeks drew all of their math, they were still very precise. The most important tools that the Greeks had were a straightedge and a compass. If we can create a length that can be made using only these tools, then we call it *constructable*. Let's see how they would draw out specific lengths.

Example 2: Whole Numbers

(i) Using your straightedge, begin by drawing a short straight line and mark their edges.

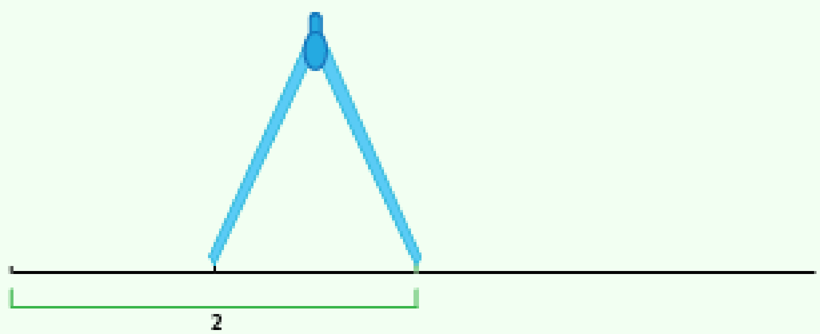
This is what we'll call our *unit length*, which means that this line is one unit long.

Example 2 Continued

(ii) Next, use your straightedge to extend the line as far as want.

(iii) We can now create the number two. Adjust your compass so that you can put one end on each of the marks that you made.

(iv) Without changing the length of the compass, shift the compass over so one end is still on a mark, and the other end is further on the line.



Create a new mark where the compass is placed on the line. You have now created the number two!

Since our straightedge is simply a ruler, we can verify that the “two” that we constructed is twice as long as the unit length simply by measuring it.

Once we’ve constructed a number once, we never have to reconstruct it. This is because we can always use our compass to measure a length that we’ve already created.

Exercise 1

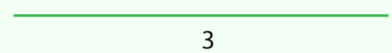
Create the number four and then construct a triangle where one side has length 2 and another has length 4.

While it may not be too difficult to create whole numbers, creating fractions (using just a straightedge and compass) is more complicated.

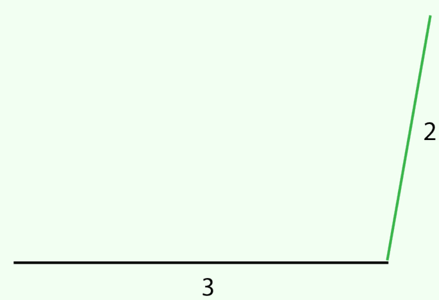
Example 3: Fractions

We'll construct the fraction $\frac{2}{3}$.

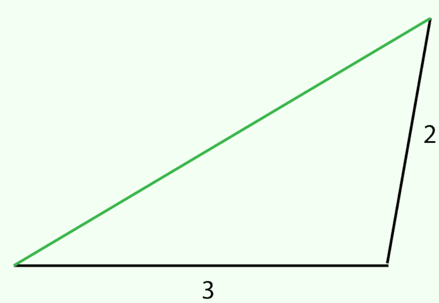
(i) Draw the length of 3



(ii) In any direction that isn't the same as the first line, draw a line of length 2 from the right end of the first line.

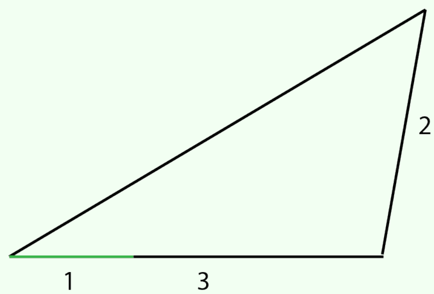


(iii) Complete the triangle

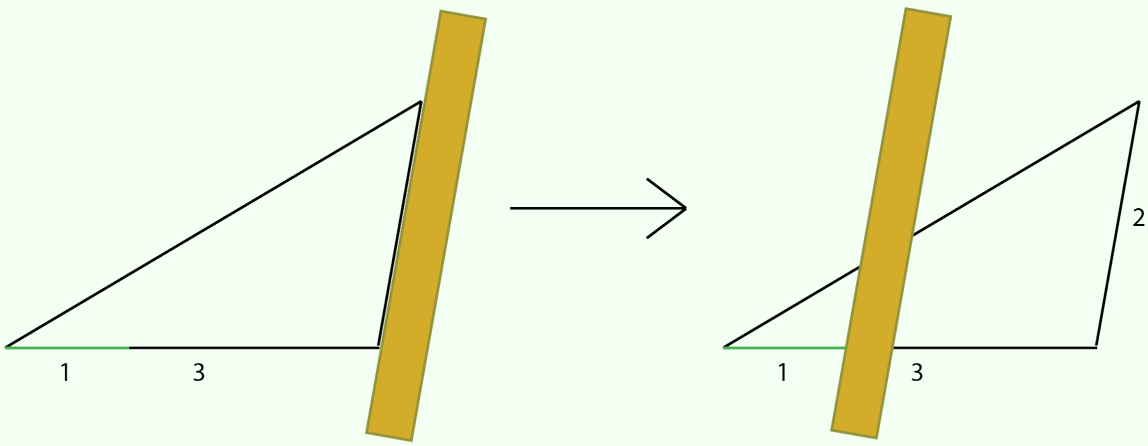


Example 3 Continued

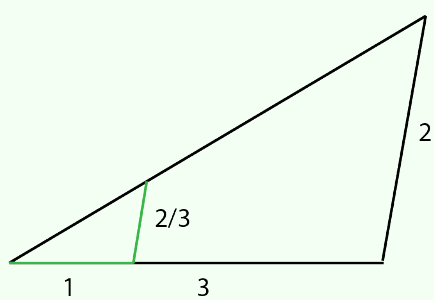
(iv) On the line of length 3, measure out a unit length from the left corner.



(v) Next, we need to draw a parallel line. To do this, line up your straightedge with the line of length 2. Carefully slide your straightedge to the right until you reach the unit length.[a]



(vi) Trace along your straightedge from the unit length to the top of the triangle. This length is $\frac{2}{3}$!



We can verify that this length is indeed $\frac{2}{3}$ by measuring it with a ruler and checking that it is $\frac{2}{3}$ the length of your unit length.

[a]Note that this is a very messy way to draw a parallel line, see the problem set to learn how to properly draw parallel lines.

Just as with whole numbers, now that we've constructed these fractions once, we no longer have to construct them again as we can just copy them.

To create any fraction $\frac{a}{b}$, we can repeat the process outlined in the example, by replacing 2 with a and 3 with b.

Exercise 2

Draw the lengths $\frac{5}{3}$ and $\frac{4}{2}$. What do you notice about the length $\frac{4}{2}$?

Hint: Compare it to other lengths that you have already made.

Rectangles and Areas

Another way that the Ancient Greeks represented numbers were as areas of rectangles.

Recall

The area of a rectangle is

$$Area = length \times width$$

Consider the following problem:

Example 4: Pizza Problems!

Sean and Thomas bought a rectangular pizza with length 6 and width $\frac{5}{3}$, but the only box they have to put the pizza in has length 2. If they cut their pizza to fit in the box, how wide must the box have been?

Many of you might be able to solve this already using algebra. Since the area of the pizza and the box should be the same, we have

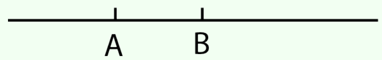
$$6 \times \frac{5}{3} = 2 \times w$$

where w is the missing width. Simplifying the left side and dividing both sides by 2, we get $w = 5$.

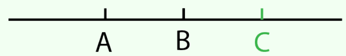
Remember that we want to solve this using only geometry. Let's learn how to draw rectangles like the Ancient Greeks. We know that rectangles have 4 right angles, so we'll start by creating right angles.

Example 5: Right Angles

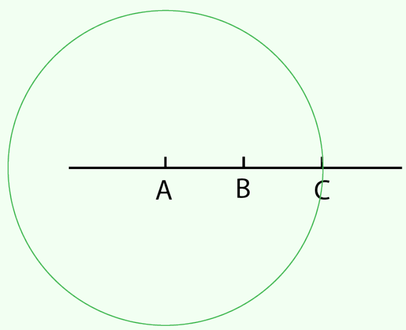
(i) Draw a straight line and mark two points, A and B.



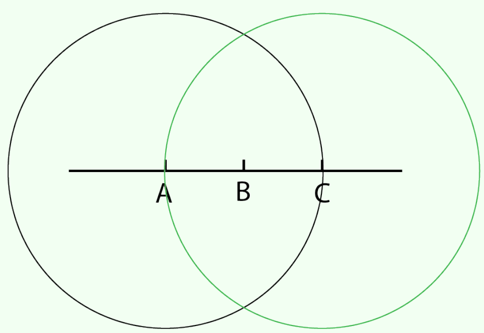
(ii) Make a third point, C, on the line that is the same distance away from B as A is (Use your compass to ensure the distance is the same)



(iii) Using your compass draw a circle centred at A with radius the length of AC.

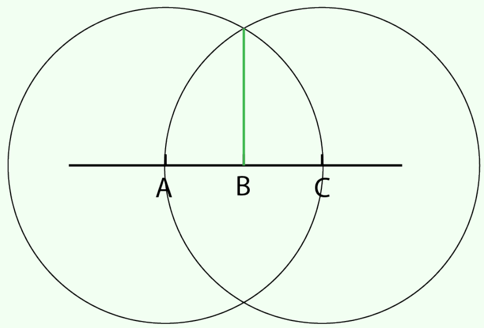


Draw a second circle centred at C with radius the length of AC as well.



Example 5 Continued

(iv) The two circles meet. Connect a point of intersection to B. The original line and this new line form a right angle!



Exercise 3

Draw a right-angled triangle where two of the lengths are 3 and 4.

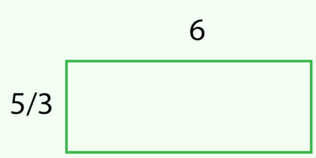
This is far too complicated to do more than a few times, so when we construct our rectangles, we will just use a protractor.

Now that we can draw rectangles, we can help Sean and Thomas. To solve for missing side lengths, we can do the following:

Example 6: Solving for Side Lengths

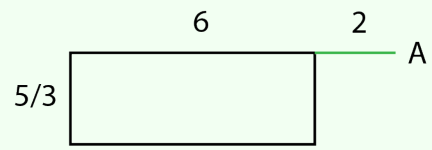
We have a rectangular pizza with length 6 and width $\frac{5}{3}$. We want to put it in a box with length 2. How wide will our box be if it has the same area as the pizza?

(i) Start by drawing the known rectangle.

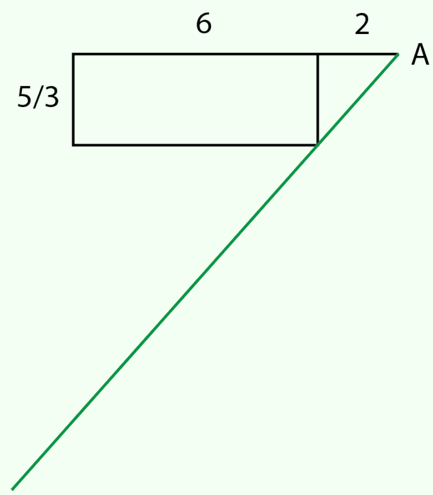


Example 6 Continued

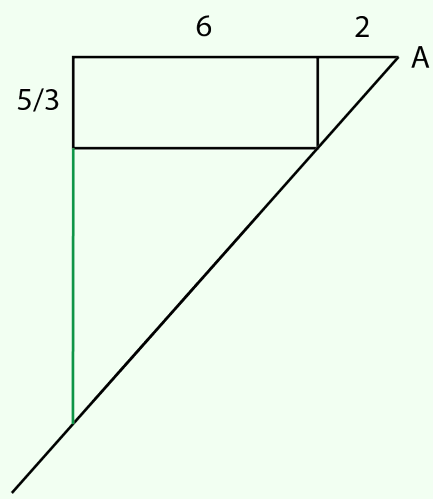
(ii) Extend the top of the rectangle by 2 (the length of the box). Label the end of the line A.



(iii) Connect the corner of the rectangle with A. Extend this line as far as you can.



(iv) Extend the far side of the rectangle straight down until you reach the diagonal line. This extension is the missing length!



Recall that we got $w = 5$ using our more familiar algebra. Verify that the length of this extension is 5 times the length of your unit length.

Exercise 4

Using the same technique as in example 6, try to solve the following problem by yourself:
Ethan and Khanh are building a shelf. They went to the hardware store and bought a piece of wood that is 4 inches wide and 7 inches long. They want to transform the wood so that it is only 2 inches wide. What will the length of their shelf be?
If you can, verify that your answer is correct using algebra.

Stop and Think

Whenever we're told something in mathematics, it's important to ask ourselves, "why is this true?" So let's stop and think about why the mysterious final line in the example is the missing length in our problems.

Congruent Triangles

Before we can justify that our answers are correct, we need to review what it means for two triangles to be congruent.

Recall

Two triangles are *congruent* if you can do one or more of the following to go from one triangle to the other:

(i) Shift the triangles,

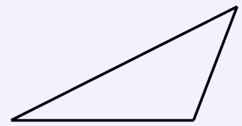
(ii) Rotate the triangles,

(iii) Reflect the triangles.

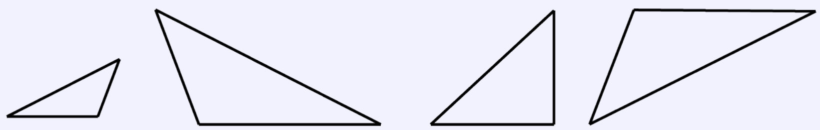
Essentially, two triangles are congruent if they look the same and are the same size.

Exercise 5

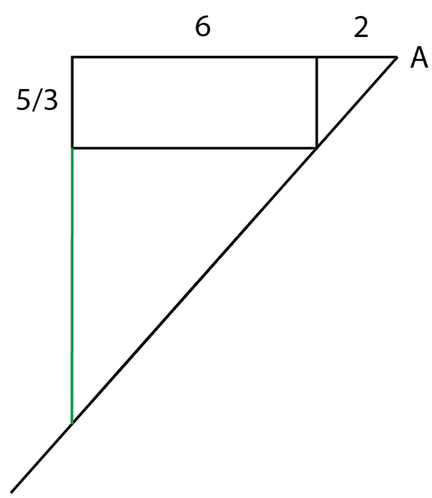
Consider the following triangle:



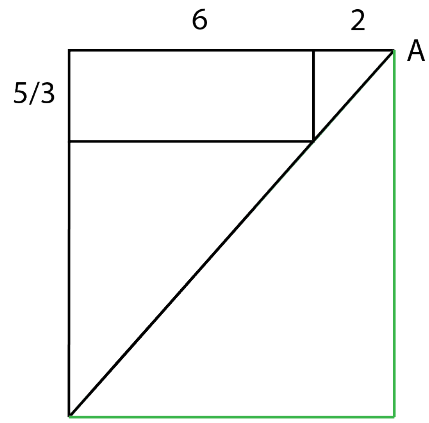
Which of the following triangles are congruent to the above triangle?



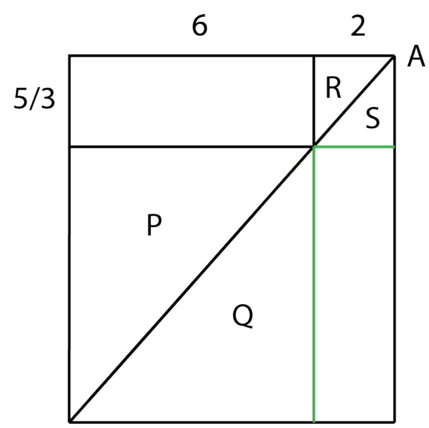
If two triangles are congruent, then they have the same areas. Now that we know a little about congruent triangles, we're ready to *justify* why our mysterious length was correct. We'll start by looking at our final step in Example 5.



Notice that we essentially have one big triangle that's being cut up into smaller pieces. Let's create a second congruent triangle and rotate it to make a large rectangle.



Since all we did is rotate the triangle, we know that the two triangles are congruent and thus have the same area. Let's divide the bottom triangle into smaller pieces and make some labels.



Notice that we've formed even more congruent triangles, namely P and Q are congruent and R and S are congruent. Since the areas of the P and Q are the same, and the areas of R and S are the same, the only missing pieces are the rectangles, so they must have the same areas!

We also have that the bottom rectangle has width 2, so it really is the area that we're looking for. So the missing length isn't so mysterious at all!