Grade 7/8 Math Circles

Week of 10th October

Solids & Platonic Solids

The History of Shapes and Solids

The general notion of shapes have been around for centuries, from the circle which was extensively worked on by ancient Greeks, to our modern day understanding of complex geometries. Since they are everywhere, we as mathematicians have come around to studying them, classifying them, and learning about all their special properties. While shapes take up our 2D world, solids are what we call 3D shapes. Solids share very similar properties to the 2D shapes, but we'll build up to that. The ancients knew much about solids as well, and even found some very special ones called the *Platonic Solids*.

Stop and Think

How many different kinds of shapes can you name? What about solids? How do you think geometers have classified shapes and solids?

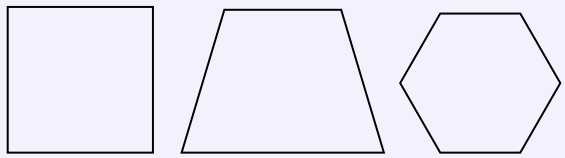
Classifying Shapes and Solids

Before we move on, let's define some common terminology!

Definitions

- Vertex: A point where two or more curves or lines meet
- Edge: A line segment joining two vertices
- Face: A flat surface (a planar region) that forms part of the boundary of a solid object
- Polygon: A plane figure made up of line segments connected to form a shape (polygons are shapes)
- Polyhedron: A three-dimensional shape with flat polygonal faces, straight edges and sharp corners or vertices (polyhedrons are solids)

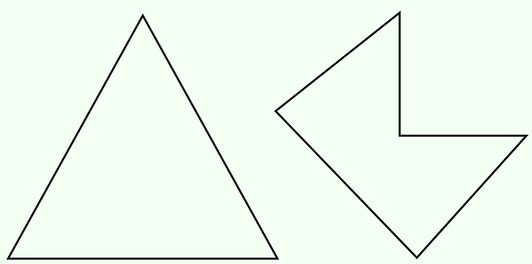
Exercise 1
Determine the number of vertices, edges, and faces each polygon has.



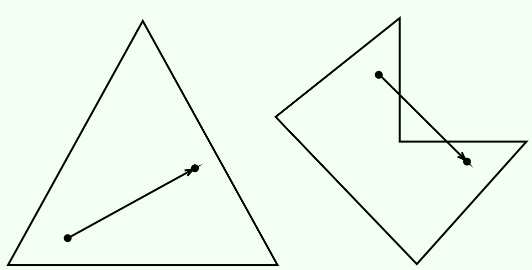
Convex and Concave Polygons and Polyhedrons

With those out of the way, let's start with our first classification; convex and concave shapes and solids. Convex polygons and polyhedrons are constructed in a way that if you chose any two points **inside** the object, you would be able to connect them with a line that stays **inside** the object. If it isn't convex, we say it's **concave**.

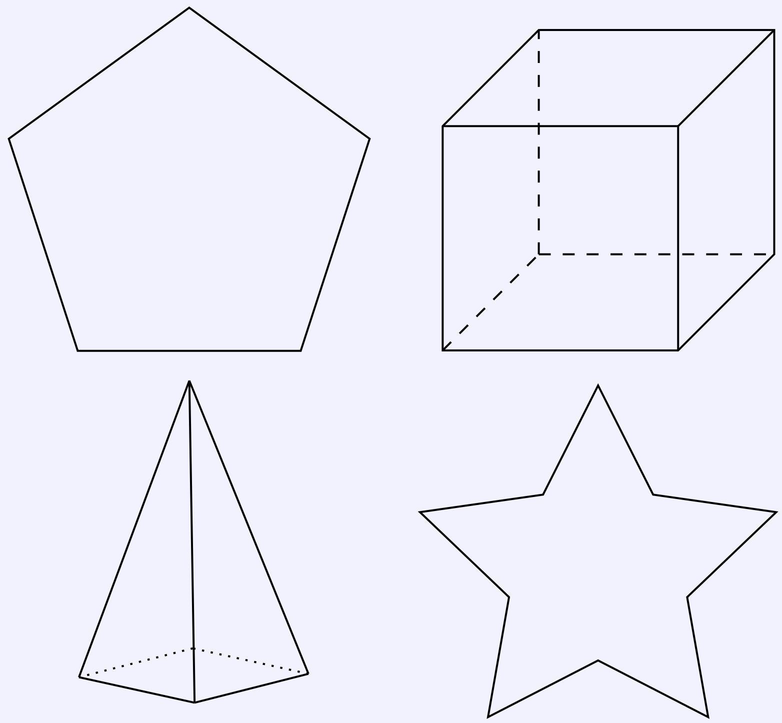
Example 1
Determine whether or not the following polygons are **convex** or **concave**.



Solution: Since we know we can check for concavity by connecting two points, we notice any two points in the triangle we draw can be connected by a line segment which stays **inside** the polygon. For the other polygon, we can find two points that cannot be connected by a line segment that stays inside. So, the triangle is a **convex** polygon, and the shape on the right is a **concave** polygon



Exercise 2
Given the following polygons and polyhedrons, determine whether they are **convex** or **concave** (as bonus, can you name all the polygons and polyhedrons shown below? How many vertices, edges, and faces do each have?)



What happens when we have a complicated polygon? What happens when you can't really tell if you can or can't connect two points with a line that stays within the boundary? Lucky for us, the Greeks proved that a polygon is convex *if and only if* no internal angle is above 180°. The term "*if and only if*" means that either both statements are true or both are false. Thus, going back to Exercise 1, we see that the all the internal angles of the pentagon do not exceed 180°, but, in the star, a few interior angles do exceed 180° (can you point out which ones?).

Stop and Think
What other ways can we check for concavity? Hint: what happens when you look at the half-plane at each edge? The half-plane is the plane created from that edge and beyond.

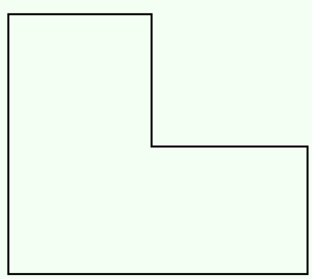
Regular and Irregular Polygons and Polyhedrons

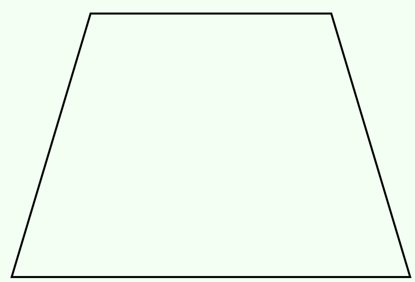
Now that we properly understand concavity, let's look at some other properties geometers like to use to classify polygons and polyhedrons. It turns out, when the angles and the edges are all the same, the shape is very nice. So, let's introduce two new terms

- Equilateral Polygon: A polygon which has all sides of the same length
- Equiangular Polygon: A polygon whose vertex angles are equal

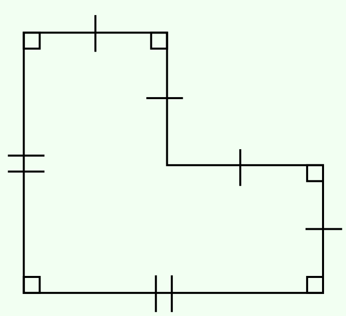
When a given polygon or polyhedron is both equilateral and equiangular, we say that the polygon is **regular** (if not, they are **irregular**). Regular polyhedra are highly symmetric and are constructed with regular polygons as their faces (this is by definition). To denote edges and angles that are the same, we use ticks; the number of ticks represents which edges/angles are the same.

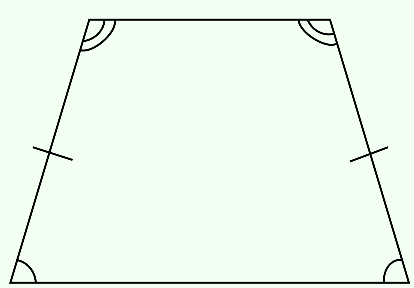
Example 2
Label all the sides and angles that are congruent (recall congruent means "the same").





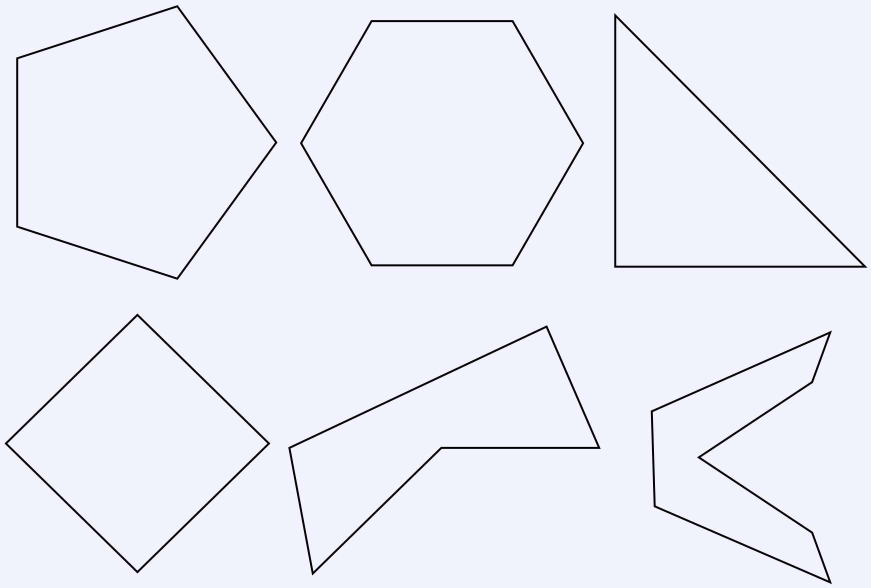
Solution: Since these polygons are sufficiently easy to work with, we can just look at them and discern the congruent angles and sides.



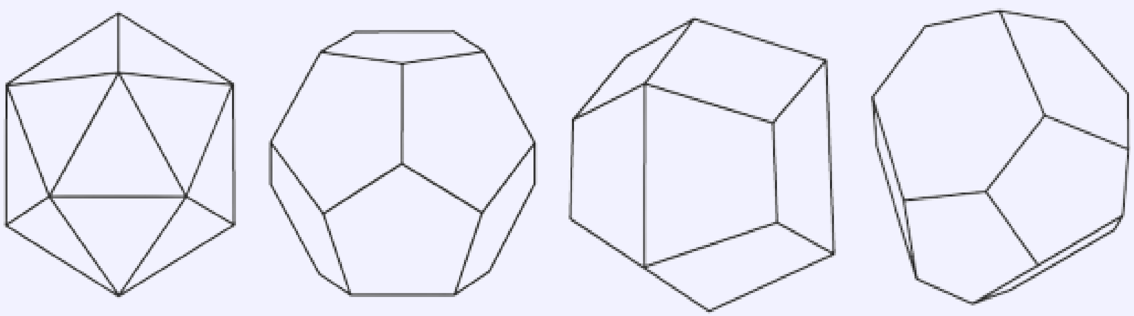


Since neither of these polygons are equilateral or equiangular, they are **irregular**.

Exercise 3
Given the following polygons, which are regular? Label each congruent angle and edge.



Exercise 4
Determine whether or not the following polyhedrons are regular (Hint: look at the faces)



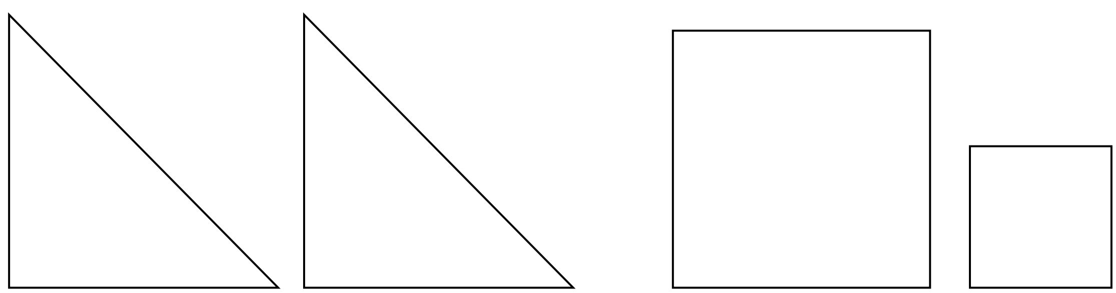
Regular polygons/polyhedrons have what's called "rotational symmetry"; more specifically, an n-gon has "order n" rotational symmetry. An n-gon is a regular polygon (equilateral and equiangular) which has n sides. This means that, using a triangle as an example, you can rotate the triangle around 3 times, and each time, it will look the exact same. You can do this 4 times with a square, 5 times with a pentagon, and so on. Another neat thing is that if you were to draw a circle around any regular polygon (or a sphere around a regular polyhedron), all the vertices would perfectly touch, and lie on, this circle (or sphere). Try it yourself! Draw any n-gon, and **circumscribe** a circle around it.

Similar and Congruent Polygons & Polyhedrons

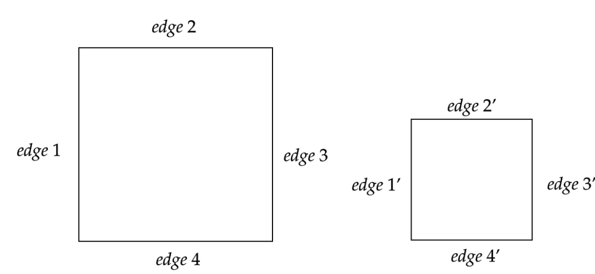
Geometers care about classifying polygons and polyhedrons, but they also care about knowing which polygons and polyhedrons are the same (or close to being the same). We introduce two new terms!

- Congruence: two objects (polygons or polyhedrons) are congruent if they have the same shape and size
- Similar: two objects (polygons or polyhedrons) are similar if they have the same shape

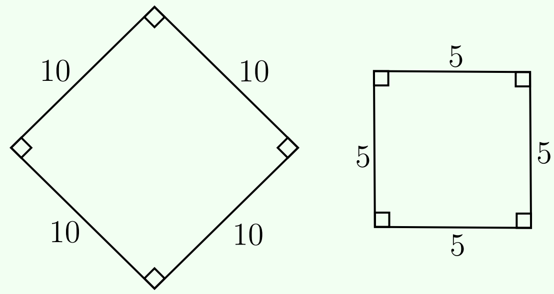
Two congruent triangles and two similar squares are shown below. The triangles are quite literally the exact same. Same edge length, and same angle measure. The two squares are similar, they are the same shape, but their edge lengths are a bit different, one is smaller!



We note that similar polygons are such that the **angles** are the same, but the **edges** are proportional to each other via some scaling factor. If you're given two similar polygons, you can start with one of them, and acquire the other by simply scaling the first polygon (with additional translation, reflection, or rotation if necessary). Note that when checking for proportionality, you take each edge in a clockwise (or counterclockwise) sequence, and check that each corresponding edge in the other polygon is proportional, and similarly for angles, take each angle in a chosen sequence and make sure the other polygon's angles are equal in measure. In the image beside this text, you check *edge 1* and *edge 1'*, and so on. Note that the proportion between edges must remain constant, they must all differ by the same scaling factor.

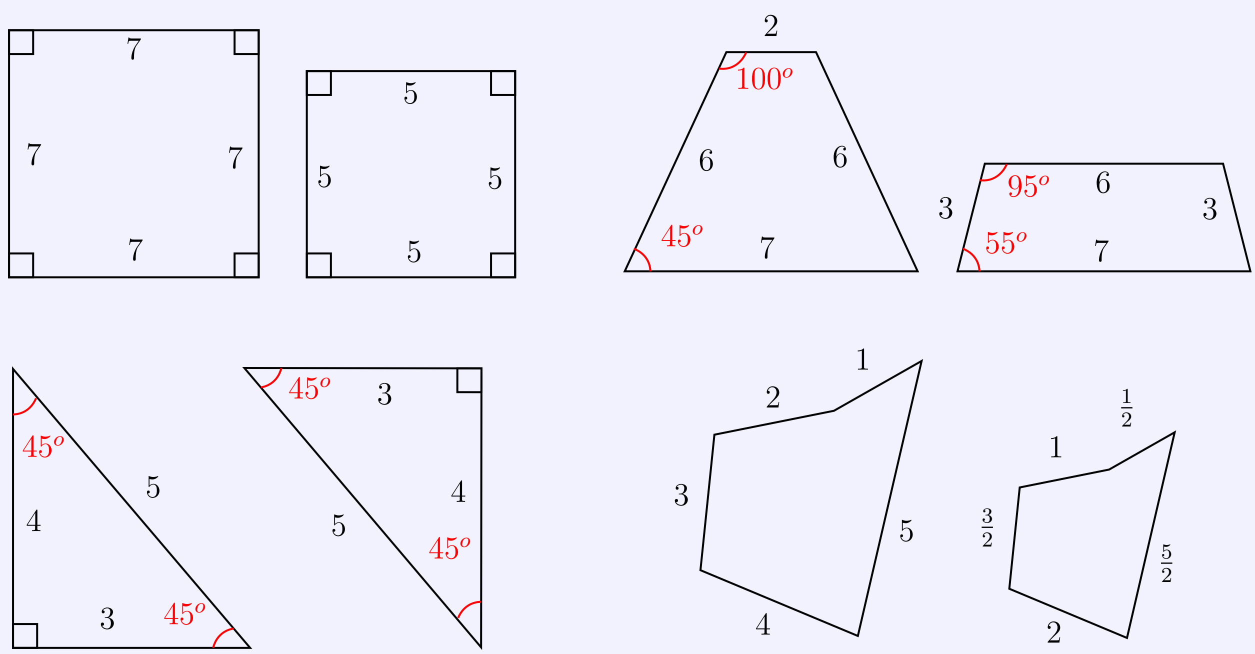


Example 3
Given these 2 polygons, determine if they are similar.



Solution: Because we know the angles are all the same (all right angles, 90^o), and all the side lengths follow the same proportionality ($2 \cdot 5 = 10$), we have that the two polygons are similar.

Exercise 5
Determine whether the following pairs of polygons are similar, congruent, or neither. (Bonus: if they are similar, can you determine the scaling factor?)



Stop and Think

If two triangles are similar, with the lengths of one triangle equal to k times the lengths of the other, how much bigger is the area of the first triangle than the area of the other? What about squares? Can you derive a general formula?

Similarity and congruence is very important in geometry with polygons and polyhedra, but it's also very important in other fields of mathematics. Fractals use notions of similarity, and even redefine our notion of dimension (but this isn't for today; if you want to read into this more, look into **fractal dimension**). When we look at other polygons, we notice something very interesting; all circles are similar to each other, all squares are similar to each other, and all equilateral triangles are similar to each other. It turns out, all **regular** n-gons are similar to each other. Recall that the definition of regular means that the polygon is **equilateral** (all edges are the same length), and **equiangular** (all angles are the same), thus, if you have a bigger or smaller polygon (has to have the same number of sides), the edges will differ by a constant proportion, and therefore are similar.

Stop and Think

What other criteria can we use to check for similarity other than the edges and the angles? (Hint: if we connect the vertices to each other, do the diagonals also share the same scaling constant?)

The Platonic Solids

Time for the epitome of this lesson. We've learnt a lot of things so far, so let's do a little recap!

Polygons and Polyhedrons Recap!

- Convex: Describes a polygon/polyhedron such that a line segment between any two points of the polygon is still inside the polygon
- Concave: Describes a polygon/polyhedron that isn't convex
- Regular: Describes a polygon/polyhedron which is equilateral and equiangular (recall equilateral and equiangular mean same edge length and angle measure respectively)
- Irregular: Describes a polygon/polyhedron that isn't regular
- Similar: Two polygons/polyhedrons are similar if they have the same shape
- Congruent: Two polygons/polyhedrons are similar if they have the same shape and size

The platonic solids require bits and pieces from all over this lesson; so let's first define and show them, then carefully explain each and every concept thoroughly. The Platonic Solids are defined as **convex**, **regular** polyhedrons in 3D space. The faces of the platonic solids are made up of **congruent** polygons. There are only 5 such polyhedra, these are shown below.

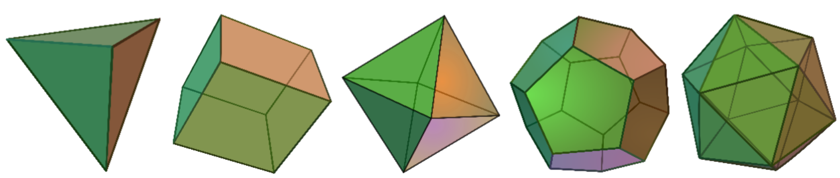


Figure 1: The 5 platonic solids

From left to right they are called the **tetrahedron**, **cube**, **octahedron**, **dodecahedron**, and the **icosahedron**.

Exercise6
Determine the number of vertices, edges, and faces each Platonic solid has. Name the polygon used to construct each polyhedron.

The Platonic solids have been known since the ancient Greeks (∼300 BCE), and archaeologists even discovered carved stone shapes from the late Neolithic people of Scotland (∼7000 BCE) which closely resemble the Platonic solids. The platonic solids are aptly named after the philosopher Plato, who used them as analogous objects for the natural elements (water, fire, earth, air, and the cosmos). Euclid then studied them extensively and dedicated a whole chapter in his famous mathematical texts, *The Elements*, to these solids and their properties. Euclid studied and consequently came up with the requirement for a convex polyhedron to be a Platonic solid. These conditions are:

I. all its faces are congruent convex regular polygons

II. none of its faces intersect except at their edges

III. the same number of faces meet at each of its vertices.

Because of this criterion, it's become common practice to denote the Platonic solids (and any other polygon, polyhedron, or polytope) using what's called the **Schläfli symbol**. This symbol is denoted $\{p, q\}$, where p is the number of edges (or, equivalently, vertices) of each face, and q is the number of faces (or, equivalently, edges) that meet at each vertex.

Exercise 7
Determine the Schläfli symbol of each Platonic solid.

Another very important property of geometry – and you'll come to learn it's incredibly important in higher level mathematics as well if you decide to study maths in university – is that well behaved polyhedrons follow this rule, Euler's formula

$$V - E + F = 2$$

Where V = number of vertices, E = number of edges, and F = number of faces. In other cases of geometry, it doesn't have to be equal to 2, and Euler actually introduced the **Euler Characteristic**, which is $\chi = V - E + F$. Various different geometries utilize this characteristic to describe the shape of surfaces. It being 2 tells us that if we were to "blow" up the platonic solids with air, they would form perfect spheres. In general, **convex polyhdra** have Euler characteristic 2.

Exercise 8
Try it out yourself. Show that Euler's formula holds for each Platonic solid.

Now, you may be asking how we know there are only 5 Platonic solids; and here's a proof (don't worry if you struggle with this proof conceptually). Since every face has p edges there would be a total of pF edges in all except that every edge is shared by two faces. Therefore $pF = 2E$. On the other hand, q edges meet at every vertex. Since each edge connects two vertices, $qV = 2E$. Therefore

$$pF = 2E = qV$$

Using this, **Schläfli symbols**, and **Euler's Characteristic**, we have that

$$\begin{aligned}
V - E + F &= 2 && \text{By Euler's characteristic}\\
\frac{2E}{q} - E + \frac{2E}{p} &= 2 && \text{By our neat fact}\\
\frac{2E}{q} + \frac{2E}{p} &= 2 + E && \text{By some algebraic manipulation}\\
\frac{1}{q} + \frac{1}{p} &= \frac{1}{E} + \frac{1}{2} && \text{dividing by } 2E \text{ on both sides}\\
\frac{1}{q} + \frac{1}{p} &> \frac{1}{2} && \text{Since } E \text{ is always a positive number}
\end{aligned}$$

We also know that both p and q must be at least 3 (can you think of why?), the only possible combinations for p and q are $\{p, q\} = \{3, 3\}, \{4, 3\}, \{3, 4\}, \{5, 3\}, \{3, 5\}$. These are exactly the Schläfli symbols for the platonic solids! Play around with this a bit more, plug in the numbers and verify yourself that these are in fact that only numbers that work.

Stop, Think, and Visualize!

Why is it that p and q must be at least 3? Take your two hands and see if you can cover an entire region of space? Flatten your hand and pretend your fingers touch at a vertex; are you able to see through? How many more hands would you need to enclose the space?