# Grade 7/8 Math Circles
## October 17/18/19/23, 2021
## Introduction to Proofs - Solutions

## Exercise Solutions

**Exercise 1**

Does the following reasoning hold true?

Hypotheses:

- If I did homework this evening, then I didn't finish it in class.
- I finished my homework in class.

Conclusion: I didn't do homework this evening.

**Exercise 1 Solution**

This argument is correct! Since I finished my homework in class, I didn't have anything do finish at home, and so I didn't do homework tonight.

**Exercise 2**

Perform the following calculations. Try to find patterns in your answers.

1. $6 + 3 =$

   $3(2 + 1) =$
2. $15 - 10 =$

   $5(3 - 2) =$
3. $4(3) + 4(2) =$

   $4(3 + 2) =$

**Exercise 2 Solution**

1. $6 + 3 = 9$

   $3(2 + 1) = 9$
2. $15 - 10 = 5$

   $5(3 - 2) = 5$
3. $4(3) + 4(2) = 20$

   $4(3 + 2) = 20$

**Exercise 3**
Now try factoring these on your own!

1. $4x - 4y + 8$
2. $2a + 6ab + 5ac$
3. $16d + 12de$

**Exercise 3 Solution**

1. $4x - 4y + 8 = 4(x - y + 2)$
2. $2a + 6ab + 5ac = a(2 + 6b + 5c)$
3. $16d + 12de = 4d(4 + 3e)$

**Exercise 4**
Now try expanding these on your own!

1. $x(x + 2)$
2. $3(a + b + 2c)$
3. $2y(x + 4)$

**Exercise 4 Solution**

1. $x(x + 2) = x^2 + 2x$
2. $3(a + b + 2c) = 3a + 3b + 6c$
3. $2y(x + 4) = 2xy + 8y$

**Exercise 5**
Prove the following implication.

If $x$ and $y$ are both even, then $x + y$ is even.

**Exercise 5 Solution**

*Proof.* We let the hypothesis be true, meaning $x = 2a$ and $y = 2b$. We want to show $x + y = 2k$ for some integer $k$. Consider $x + y$.

$$\begin{aligned} x + y &= 2a + 2b \\ &= 2(a + b) && \text{(common factor a 2)} \\ &= 2k && \text{(where } k = a + b\text{)} \end{aligned}$$

Since $x + y = 2k$ for some integer $k$, $x + y$ is even!

□

**Exercise 6**
Disprove the following statement. If $x$ and $y$ are both odd numbers, then $x^2 + 3y$ is odd.

**Exercise 6 Solution**

*Proof.* We claim that this is false. A counter-example is $x = 3$ and $y = 5$.
$x$ and $y$ are both odd numbers so the hypothesis is true. But $x^2 + 3y = 24$ which is even so the conclusion is false.
Therefore, the statement is false.

□

**Exercise 7**
Let A "I scored a goal." Let B = "I get a McDonalds meal."

(a) Write the implication A $\implies$ B in words.

(b) Write the converse of A $\implies$ B in words.

**Exercise 7 Solution**

(a) If I scored a goal, then I get a McDonalds meal.

(b) If I get a McDonalds meal, then I scored a goal.

**Exercise 8**
In Exercise 5, you proved that if $x$ and $y$ are both even numbers, then $x + y$ is even. Now let's focus on the converse of this statement.

(a) Write the converse of this statement in words.

(b) If you think the converse is true, prove it!. If you think it is false, give a counter-example!

**Exercise 8 Solution**

(a) The converse is, "if $x + y$ is even, then $x$ and $y$ are both even."

(b) *Proof.* We claim converse is *false*!

A counter-example is $x = 7$ and $y = 9$. Note that $x + y = 16$ which is even, so the hypothesis is true.

However, $x$ and $y$ are odd, so the conclusion is false.

Therefore, the converse is false. □

**Exercise 9**
Write the following numbers in expanded form.

(a) 62

(b) 389

**Exercise 9 Solution**

(a) $62 = 10(6) + 2$

(b) $389 = 100(3) + 10(8) + 9$

**Exercise 10**
Determine if the using following number are divisible by 3 using the rule from Example 10.

(a) Is 84 divisible by 3?

(b) Is 186 divisible by 3?

(c) Is 4592 divisible by 3?

(d) Write down a five-digit number that is divisible by 3.

**Exercise 10 Solution**

(a) Yes. The sum of the digits of 84 is $8 + 4 = 12$ and 12 is divisible by 3.

(b) Yes. $1 + 8 + 6 = 15$ and 15 is divisible by 3.

(c) No. $4 + 5 + 9 + 2 = 20$ and 20 is not divisible by 3.

(d) There are *plenty* of examples. Here are a few: 12345, 96720, and 23457.

## Problem Set Solutions

1. State the hypothesis and conclusion of the following statements.

   (a) If $x$ is positive, then $x - 7$ is positive.

   (b) If I walk to school today, then Jacob takes the bus to school today.

   *Solution*:

   (a) Hypothesis: $x$ is positive. Conclusion: $x - 7$ is positive.

   (b) Hypothesis: I walk to school today. Conclusion: Jacob takes the bus to school today.

2. Determine why this proof does not hold true.

   Hypotheses:

   - John is responsible for walking his dog at least twice per week.
   - John didn't walk his dog on Monday, Tuesday, or Wednesday.

   Conclusion: Therefore John walked his dog on Friday.

   *Solution*: We can find a counter-example. Let's say John walked his dog only on Thursday and Saturday this week. Both of the hypotheses are satisfied; he walked his dog at least twice this week and didn't walk his dog on Monday, Tuesday, or Wednesday. However, the conclusion is false. He did not walk his dog on Friday. Therefore, the argument is false.

3. Disprove the following statements.

   (a) If $x$ is positive, then $x - 10$ is negative.

   (b) If $x$ is positive, then $x - 10$ is positive.

   (c) If $x$ and $y$ are even, and $z$ is odd, then $x + y + 2z$ is odd.

*Solution*:

(a) A counter-example is $x = 15$ because 15 is positive but $15 - 10 = 5$ which is positive, not negative.

(b) A counter-example is $x = 5$ because 5 is positive but $5 - 10 = -5$ which is negative, not positive.

(c) A counter-example is $x = 4$, $y = 6$, and $z = 3$ because $4 + 6 + 2(3) = 16$ which is even, not odd.

4. Fully factor the following expressions.

(a) $15x + 9xy$

(b) $10a + 2b - 8$

(c) $x^2 + x$

*Solution*:

(a) $15x + 9xy = 3x(5 + 3y)$

(b) $10a + 2b - 8 = 2(5a + b - 4)$

(c) $x^2 + x = x(x + 1)$

5. Expand the following expressions.

(a) $3(10x + 9y - 4)$

(b) $5a(2a + 3b)$

(c) $2xy(5x + 1)$

*Solution*:

(a) $3(10x + 9y - 4) = 30x + 27y - 12$

(b) $5a(2a + 3b) = 10a^2 + 15ab$

(c) $2xy(5x + 1) = 10x^2y + 2xy$

6. Prove the following statements.

(a) If $x$ is even and $y$ is odd, then $xy$ is even.

(b) If $x$ is even and $y$ is odd, then $x + y$ is odd.

(c) If $x$, $y$, and $z$ are all odd, then $x + y + z$ is odd.

*Solution*:

(a) *Proof.* Assume the hypothesis is true. So, $x = 2a$ and $y = 2b + 1$ for some integers $a$ and $b$. Consider $xy$.

$$\begin{aligned} xy &= (2a)(2b+1) \\ &= 4ab + 2a && \text{(distribute } 2a\text{)} \\ &= 2(2ab + a) && \text{(common factor a 2)} \\ &= 2k && \text{(where } k = 2ab + a\text{)} \end{aligned}$$

Since $xy = 2k$ for some integer $k$, $xy$ is even. □

(b) *Proof.* Assume the hypothesis is true. So, $x$ is even and $y$ is odd. We need to show $x + y = 2k + 1$ for some integer $k$.

Since $x$ is even and $y$ is odd, we can write $x = 2a$ and $y = 2b + 1$ for some integers $a$ and $b$. Consider $x + y$.

$$\begin{aligned} x + y &= 2a + 2b + 1 \\ &= 2(a + b) + 1 && \text{(common factor a 2)} \\ &= 2k + 1 && \text{(where } k = a + b\text{)} \end{aligned}$$

Since $x + y = 2k + 1$ for some integer $k$, $x + y$ is odd!

□

(c) *Proof.* Assume the hypothesis is true. So, $x = 2a + 1$, $y = 2b + 1$, and $z = 2c + 1$ for some integers $a$, $b$, and $c$. Consider $x + y + z$.

$$\begin{aligned} x + y + z &= 2a + 1 + 2b + 1 + 2c + 1 \\ &= 2a + 2b + 2c + 2 + 1 \\ &= 2(a + b + c + 1) + 1 && \text{(common factor a 2)} \\ &= 2k + 1 && \text{(where } k = a + b + c + 1\text{)} \end{aligned}$$

Since $x + y + z = 2k + 1$ for some integer $k$, $x + y + z$ is odd.

□

7. Prove the following statement. Is the converse true? If it is true, prove it. If it is false, give a counter-example.

   If both $x$ and $y$ are even, then $xy$ is even.

*Solution*:

*Proof.* Assume the hypothesis is true. So, $x = 2a$ and $y = 2b$ for some integers $a$ and $b$. Consider $xy$.

$$\begin{aligned} xy &= (2a)(2b) \\ &= 4ab \\ &= 2(2ab) && \text{(factor a 2)} \\ &= 2k && \text{(where } k = 2ab\text{)} \end{aligned}$$

Since $xy = 2k$ for some integer $k$, $xy$ is even.

□

*Proof.* Now consider the converse.
If $xy$ is even, then both $x$ and $y$ are even. We claim this is false and below is a counter-example.
Consider $x = 5$ and $y = 4$. The hypothesis is true because $xy = 20$ which is even. However, the conclusion is false because $x = 5$ is odd, not even.

□

8. CHALLENGE PROBLEM 1

Prove the divisibility by 3 rule for three-digit numbers. That is, prove the following statement.

A three-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

HINT: The proof from Example 10 in the lesson is a good guide for this question.

*Solution*:

*Proof.* Let $x$ be a three-digit number with digits $a$, $b$, and $c$. Then $x = 100a + 10b + c$ because $a$ is the hundreds digit, $b$ is the tens digit, and $c$ is the ones digit.
$\implies$ ("If $x$ is divisible by 3, then the sum of its digits is divisible by 3.")
Since $x$ is divisible by 3, we can find an integer $k$ such that $x = 3k$. Therefore,

$$\begin{aligned} 100a + 10b + c &= 3k & \\ a + b + c &= 3k - 99a - 9b & \text{(subtract } (99a + 9b) \text{ from both sides)} \\ a + b + c &= 3(k - 33a - 3b) & \text{(common factor a 3)} \\ a + b + c &= 3m & \text{(where } m = k - 33a - 3b) \end{aligned}$$

Therefore, $a + b + c$, which is the sum of the digits of $x$, is divisible by 3.
$\Longleftarrow$ ("If the sum of the digits of $x$ is divisible by 3, then $x$ is divisible by 3.")
Since the sum of the digits of $x$ is divisible by 3, $a + b + c = 3k$ for some integer $k$.

$$\begin{aligned} a + b + c &= 3k & \\ 100a + 100b + 100c &= 300k & \text{(multiply both sides by 100)} \\ 100a + 10b + c &= 300k - 90b - 99c & \text{(subtract } (90b + 99c) \text{ from both sides)} \\ 100a + 10b + c &= 3(100k - 30b - 33c) & \text{(common factor a 3)} \\ 100a + 10b + c &= 3m & \text{(where } m = 100k - 30b - 33c) \\ x &= 3m & \text{(since } x = 100a + 10b + c) \end{aligned}$$

Therefore, $x$ is divisible by 3.
Both directions of the if and only if statement have been proved, so the statement is true!

$\square$

9. CHALLENGE PROBLEM 2

This final question is a proof of a very famous theorem, but you will need to learn more about the distributive property to complete this challenge problem. Below is a mini-lesson to help with this.

**FOIL mini-lesson**

We need to learn to expand an expression of the form $(a+b)(c+d)$ and will do so with the help of the acronym FOIL. This is a step up from before where we expanded $a(b+c)$. FOIL stands for First, Outside, Inside, Last.

- First: Multiply the first term in each set of brackets. So, $ac$.
- Outside: Multiply the outside terms of the brackets. So, $ad$.
- Inside: Multiply the inside terms of the brackets. So, $bc$.
- Last: Multiply the last term in each set of brackets. So, $bd$.
- That add them all together. So, $(a+b)(c+d) = ac + ad + bc + bd$.

Here are some concrete examples.

(a)

$$\begin{aligned}(3+2)(1+5) &= (5)(6) && \text{(using BEDAMS)}\\ &= 30\\ (3+2)(1+5) &= (3)(1)+(3)(5)+(2)(1)+(2)(5) && \text{(using FOIL)}\\ &= 3+15+2+10\\ &= 30\end{aligned}$$

(b)

$$\begin{aligned}(a+2)(a+b) &= (a)(a)+(a)(b)+(2)(a)+(2)(b) && \text{(using FOIL)}\\ &= a^2+ab+2a+2b\end{aligned}$$

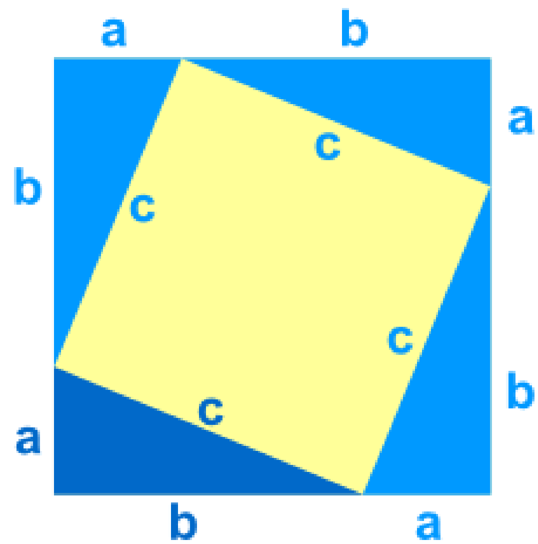


Figure 1: Image retrieved from Math is Fun

Now you are ready for the proof! We need to think about the area of the large square in two ways.

(a) The side length of the large square is $a + b$. The area of a square is the side length mutliplied by itself. So, area $= (a + b)(a + b)$.

Expand $(a + b)(a + b)$.

(b) The area of the large square can also be thought of as the area of the yellow square plus the area of the four triangles.

Recall that the area of a triangle is $\frac{(\text{base}) \times (\text{height})}{2}$.

Write the area of the large square as the area of the yellow square plus the area of the four triangles.

(c) Your answer to part (a) and part (b) are measure the same area, so they are equal! Set them equal to each other.

Then try to show that $a^2 + b^2 = c^2$

Congratulations! You just *proved* the Pythagorean Theorem!

Here's a slider made in GeoGebra that demonstrates the Pythagorean Theorem! https://www.geogebra.org/m/afygf9dq.

*Solution*:

(a) $(a+b)(a+b) = a^2 + 2ab + b^2$

(b) The area of the yellow square is $c \times c = c^2$.

The area of the one triangle is $\dfrac{(a)(b)}{2}$ so the area of four triangles is $4 \times \dfrac{(a)(b)}{2} = 2ab$.

So, the area of the full square is $c^2 + 2ab$.

(c)

$$a^2 + 2ab + b^2 = c^2 + 2ab \qquad \text{(equate the answers from part (a) and part (b))}$$
$$a^2 + b^2 = c^2 \qquad \text{(subtract } 2ab \text{ from both sides)}$$

This *proves* the Pythagorean Theorem!