



## Grade 6 Math Circles

Week of 20<sup>th</sup> November

### Matrices

Mathematicians have worked extensively with numbers, and have found ways of working with more than just 1 number at a time. When working with word problems, system of equations, and linear algebra, matrices offer mathematicians a way to describe more than one idea, problem, or quantity, and use similar math to resolve problems. For example, take the following scenario

*You have 10 blue balls and 4 red balls while your friend has 5 blue balls and 9 red balls*

Like we do with many things, we can make a table representing this information

	You	Friend
Blue Balls	10	5
Red Balls	4	9

We can take the arrangement of numbers, and put them into a matrix! We do so by taking the numbers in their exact same position, and enclosing them with a box

$$\begin{bmatrix} 10 & 5 \\ 4 & 9 \end{bmatrix}$$

Although this doesn't seem very useful to us right now, we'll quickly learn that matrices greatly simplify certain problems.

#### Exercise 1

Suppose James tutors 6 grade nine students and 10 grade ten students and Sean has 10 grade nine students and 7 grade ten students. Represent this information in a matrix.

### Matrix Dimensions

Since matrices are mathematical objects, we need to understand their properties, and make sure they follows specific rules. Dimensions of matrices are very important, and all matrix operations depend on the dimension of the matrix. The **dimension** of a matrix is denoted " $m \times n$ ", where  $m$  is the



number of *rows* and  $n$  is the number of *columns*. So, in our example above, we had a  $2 \times 2$  matrix. In the example below, we have a  $3 \times 2$  matrix;

$$M = \begin{matrix} \left[ \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array} \right] \\ \underbrace{\hspace{1.5cm}} \\ 2 \text{ columns} \end{matrix} \left. \vphantom{\begin{matrix} \left[ \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array} \right] \\ \underbrace{\hspace{1.5cm}} \\ 2 \text{ columns} \end{matrix}} \right\} 3 \text{ rows}$$

We call a matrix a *square matrix* if  $m = n$ .

### Exercise 2

Determine the dimension of the following matrices

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 & 3 \\ -1 & 0 & 8 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 & 6 \\ 4 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

## Matrix Operations

Just like with numbers, we can work with matrices using the same operations. To start, with addition or subtraction, we simply add or subtract the matrix elements that are in the same spot. That is, if we have two  $2 \times 2$  matrices  $A$  and  $B$ ,  $A + B$  and  $A - B$  are computed as follows as follows

$$A + B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix}$$

$$A - B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a - e & b - f \\ c - g & d - h \end{bmatrix}$$

**Exercise 3**

Compute the following matrix additions and subtractions

a) 
$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 4 & 3 \end{bmatrix} =$$

b) 
$$\begin{bmatrix} 3 & 12 \\ -\frac{1}{2} & 1 \end{bmatrix} - \begin{bmatrix} 1 & 7 \\ -4 & 3 \end{bmatrix} =$$

c) 
$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + \left( \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \right) =$$

Although we can multiply matrices themselves, in this lesson we'll just talk about **scalar multiplication**. First, we must understand what **scalar** means. A scalar is just a number, which means that we are trying to multiply a matrix by a number. Scalar multiplication is an operation which multiplies each element of the matrix by a number, as follows:

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

**Exercise 4**

Compute the following scalar multiplications

a) 
$$10 \begin{bmatrix} 2 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$$

b) 
$$3 \left( \begin{bmatrix} 3 & 2 \\ -\frac{1}{2} & 1 \end{bmatrix} - \begin{bmatrix} 1 & 7 \\ -4 & 3 \end{bmatrix} \right)$$

c) 
$$0 \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

The **transpose** of a matrix is an operation which doesn't change anything about the matrix other than the dimension. When you transpose a matrix, you swap the rows and the column indices of the



matrix element. For example,

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$
$$A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

Going back to our matrix elements, we have that when we transpose a matrix,  $(a_{i,j})^T = a_{j,i}$

### Exercise 5

Determine  $A^T$

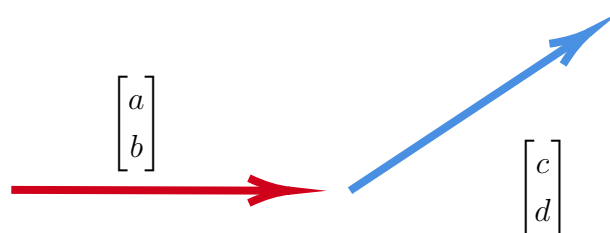
a)  $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$

b)  $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

c)  $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

## Application of Matrices

Matrices are very closely related to vectors. In fact, a vector is just an  $n \times 1$  matrix.



Matrix-Matrix multiplication is a whole process, but Matrix-Vector multiplication is a bit more



intuitive to understand. Say you are given a  $2 \times 2$  matrix  $A$  and a  $2 \times 1$  vector  $\vec{v}$ , then we can write

$$A\vec{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

This operation allows us to modify vectors with various transformations. Let us grab some matrices and see what they do to our vectors. Let's keep the arbitrary vector  $\vec{v}$ , and apply the following matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies A\vec{v} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 0 \cdot y \\ 0 \cdot x + 1 \cdot y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This matrix,  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is called the identity matrix,  $\mathbb{I}$ , and it acts like the number 1, as in  $\mathbb{I} \times A = A$ , for any matrix  $A$ . Consider the next matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \implies A\vec{v} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \cdot x + 0 \cdot y \\ 0 \cdot x + b \cdot y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

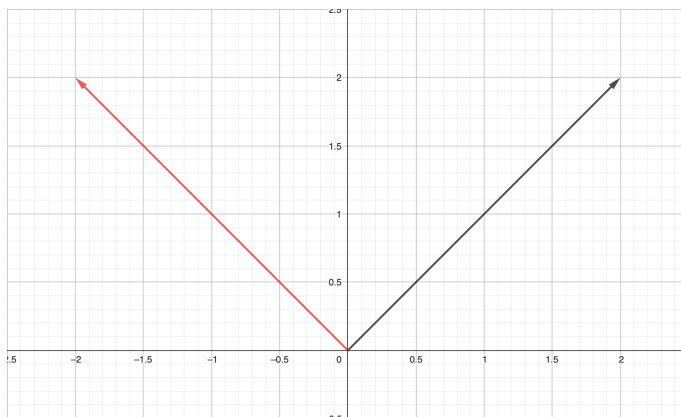
This matrix allows you to scale any vector's components individually. We call this the scaling matrix. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \implies A\vec{v} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \cdot x + 1 \cdot y \\ 1 \cdot x + 0 \cdot y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

This matrix takes your vector, and reflects it across the line  $y = x$ . It takes the  $x$  and  $y$  components, and switches them. This is a new type of matrix, as it actually manipulates the position of the components. Normally, when the matrix has elements along its diagonal, it behaves nicely when multiplying, if not, it can change the vector drastically. Finally, we have our last matrix,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies A\vec{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \cdot x + (-1) \cdot y \\ 1 \cdot x + 0 \cdot y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

This matrix actually rotates your vector  $90^\circ$  counter clockwise. Below is a diagram depicting the matrix acting on the vector  $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ .



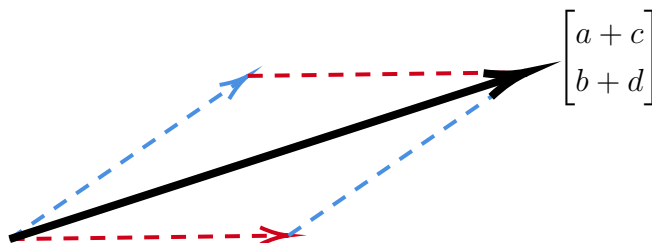
The black vector is the original vector, and the red vector is the transformed vector. Notice that

$$\begin{aligned} A\vec{v} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot 2 + (-1) \cdot 2 \\ 1 \cdot 2 + 0 \cdot 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 2 \end{bmatrix} \end{aligned}$$

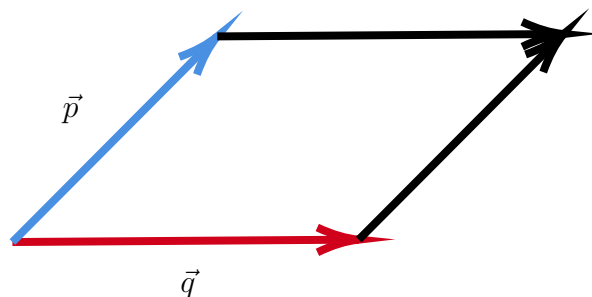
The math checks out with what we see geometrically with the vectors!

## Area of Parallelogram

When we have two vectors, when we add them, we end up – visually – with a parallelogram. Recall how we add two vectors



As we learnt in the previous lesson, we can add either vector **tip to tail**. When we do this, we see a parallelogram. In fact, we can always describe a parallelogram using a matrix! We take the two sides of the parallelogram, which are represented by two vectors, say  $\vec{p}$  and  $\vec{q}$ ,



We can put both  $\vec{p}$  and  $\vec{q}$  into a matrix, like so

$$\begin{bmatrix} \vec{p} & \vec{q} \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Once we do this, it's actually possible to find the area of the parallelogram! To do so, we must find the absolute value of the **determinant** of this matrix. The determinant of a  $2 \times 2$  matrix can be found as follows

$$\det \left( \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right) = ad - bc$$

Therefore, since we take the absolute value, we have that

$$\text{Area} = \left| \det \left( \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right) \right|$$

We can actually prove this geometrically, first, we'll draw a diagram

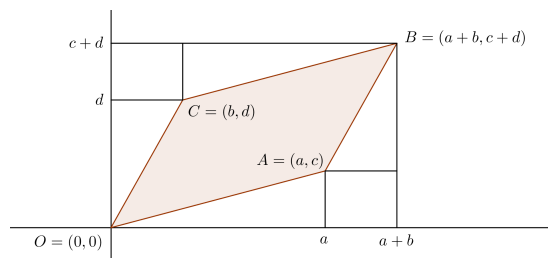


Image retrieved from [ProofWiki](#)



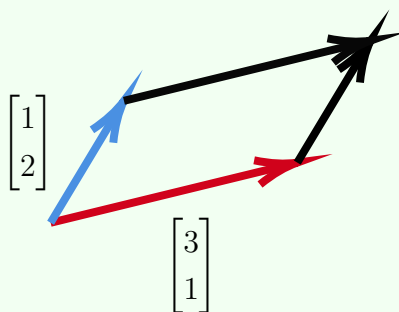
The total area of the square is  $A_{total} = (a + b)(c + d)$ . Since we only want the area shaded in red, we have to subtract the two squares and four triangles. The squares each have area  $bc$ , and the triangles have areas  $\frac{ac}{2}$  and  $\frac{bd}{2}$ . Thus, we have that

$$\begin{aligned} Area_{red} &= (a + b)(c + d) - \left(\frac{ac}{2}\right) - \left(\frac{ac}{2}\right) - \left(\frac{bd}{2}\right) - \left(\frac{bd}{2}\right) - bc - bc \\ &= ad - bc \end{aligned}$$

Which is exactly the definition of the determinant of the two vectors!

### Example 1

Say we are given the parallelogram defined via the two vectors  $\vec{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , and  $\vec{q} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



Thus, the area of the parallelogram is therefore

$$\begin{aligned} \text{Area} &= \left| \det \left( \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right) \right| = \left| \det \left( \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \right) \right| \\ &= |1 \cdot 1 - 2 \cdot 3| \\ &= |1 - 6| \\ &= |-5| \\ &= 5 \text{ units}^2 \end{aligned}$$



**Exercise 6**

Determine the area of the parallelograms defined by the following vectors

a)  $\vec{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\vec{q} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

b)  $\vec{p} = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix}$  and  $\vec{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

c)  $\vec{p} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$  and  $\vec{q} = \begin{bmatrix} -1 \\ 1/5 \end{bmatrix}$

d)  $\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\vec{q} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$